Projective varieties with ample cotangent bundle

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Table des matières

Introduction

One can obtain many information on the geometry of an algebraic variety by understanding the curves it contains. For example, the study of rational curves is a central tool in the birational classification of algebraic varieties. More generally, entering the non algebraic world, one can wonder what are the behaviors of entire curves in an algebraic variety. Let X be a smooth projective complex variety. An *e*ntire curve of X is a non-constant entire map $f : \mathbb{C} \to X$. In this direction, several questions arise naturally.

1. Does X contain any entire curve?
2. Is there a proper algebraic subset $Y \subseteq X$ such that every entire curves of X is contained in Y?
3. For any entire curve of X, is there a proper algebraic subset $Y \subset X$ containing it?

Obviously a negative answer to the first question implies a positive answer to the second question, which in turn implies a positive answer to the third question. We say that X is *hyperbolic* if it contains no entire curves.

It is in general a very difficult question to determine whether a variety is hyperbolic or not. There are several major conjectures motivating the research in this field, we state two of them.

Conjecture (Kobayashi)**.** *A generic hypersurface of high degree in projective space is hyperbolic.*

Conjecture (Green-Griffiths)**.** *A variety of general type X contains a proper algebraic subset containing all the entire curves in X.*

The central goal of this thesis is to study hyperbolicity related problems for complete intersection varieties. The technics we use mainly come from the theory of jet differentials. However there are other very successful technics. We mention the existence of important technics coming from Nevanlinna theory and from foliation theory, however we will not use those in our work.

It is possible to use algebraic-geometric tools to study the entire curves (which are typically non-algebraic objects) contained in an algebraic variety. One way of doing this is via jet differentials. This idea goes back to the work of Bloch [Blo26], but the theory was really developed in the work of Green and Griffiths. In [GG80], for any given variety X they introduce what is now called the *Green-Griffiths jet differential bundles* $E^{GG}_{k,m}\Omega_X$ which are, roughly speaking, bundles of differential equations of order k and degree m. More precisely at each point $x \in X$ the fiber $E^{GG}_{k,m}\Omega_{X,x}$ consists exactly of polynomials of the form

$$P\left(f', \dots, f^{(k)}\right) = \sum_{\substack{I_1,\dots,I_k \in \mathbb{N}^n \\ |I_1|+2|I_2|+\cdots+k|I_k|=m}} a_{I_1,\dots,I_k} (f')^{I_1} \cdots (f^{(k)})^{I_k},$$

where $n = \dim X$ and where $\left(f', \dots, f^{(k)}\right)$ is the k-jet of a germ of holomorphic curve $f : (\mathbb{C}, 0) \to X$. To make this expression precise, we should look at local coordinates around x, write $f = (f_1, \dots, f_n)$ and use the standard multi-index notation for $I = (i_1, \dots, i_n)$ and $1 \leqslant j \leqslant k$ we set $\left(f^{(j)}\right)^I := \left(f_1^{(j)}\right)^{i_1} \cdots \left(f_n^{(j)}\right)^{i_n}$. The importance of those bundles lies mainly in their applications to hyperbolicity problems via a vanishing criterion. This vanishing criterion asserts that: *Any global jet differential equation vanishing on some ample divisor has to be satisfied by every entire curve.*

Afterwards, Demailly [Dem97a] introduced a more refined version of those bundles, which are now called the *Demailly-Semple jet differential bundles* or *invariant jet differential bundles*. His idea is that, if one wants to study the entire curves in a projective variety, then the relevant information is just the geometric locus of the curves and not the way the curves are parametrized. Therefore Demailly considered the subbundle $E_{k,m}\Omega_X$ of $E_{k,m}^{GG}\Omega_X$ consisting of elements invariant under reparametrisation of the jets of curves. Certainly the vanishing criterion also holds for this bundle. In view of the vanishing criterion it has become a central problem to construct nonzero elements of $H^0(X, E_{k,m}^{GG}\Omega_X \otimes A^{-1})$ (or $H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$) for some ample line bundle on X.

As an illustration, recently, using the invariant jet differential approach, based on a strategy of Siu [Siu04], Diverio, Merker and Rousseau [DMR10] made a major breakthrough by proving effective algebraic degeneracy for entire curves in a generic hypersurface of high degree in projective space.

A particular case of those technics concerns the positivity of the cotangent bundle. More precisely, if X is a projective variety such that the cotangent bundle of X is ample, then X is hyperbolic. However the ampleness of the cotangent bundle is an extremely strong condition. Varieties with ample cotangent bundle are supposed to be abundant however relatively few examples are known. The construction of such examples is one of the main goals of this thesis, motivated by the following conjecture of Debarre [Deb05].

Conjecture (Debarre)**.** *If $X \subset \mathbb{P}^N$ is a generic complete intersection of sufficiently high multidegree such that* $\operatorname{codim}_{\mathbb{P}^N}(X) \geqslant \dim(X)$*, then the cotangent bundle of X is ample.*

We give several different partial results towards this conjecture, in particular we prove it in the case of surfaces. It is also interesting to understand complete intersections of lower codimension. One approach, cohomological, is suggested by Debarre. The other approach concerns jet differentials. With this point of view Diverio and Trapani [DT09] made a precise conjecture which somehow interpolates between Kobayashi's conjecture and Debarre's conjecture. Later on, we will point out a first evidence towards this conjecture.

Conjecture (Diverio-Trapani)**.** *If $X \subset \mathbb{P}^N$ is the intersection of at least $\frac{N}{k+1}$ generic hypersurfaces of sufficiently high degree, then $E_{k,m}\Omega_X$ is ample for all $m > 0$.*

Summary of the results

We now describe the content of this thesis.

We start in Chapter 1 with preliminaries on positivity notions for line bundles and divisors, recalling the main definitions and the principal properties. Then, we recall how to extend those definitions to vector bundles of higher rank, pointing out some of the difficulties appearing. After that, we introduce the main definitions concerning hyperbolicity, jet spaces, and jet differential equations. We will then discuss varieties with ample cotangent bundle, and explain Debarre's conjecture which is our guiding motivation during this thesis. We also discuss some arithmetical motivations.

Chapter 2 is dedicated to intersection computations on complete intersection varieties. Those computations are essential for several of our results, this is why we group them together in this chapter. We will systematically use Segre classes instead of Chern classes. From a theoretical point of view, this is certainly equivalent, but the use of Segre classes will highly simplify some of our computations. We start by studying how intersection computations in the Demailly jet tower can be deduced from intersection computations on the given base variety. Then we detail intersection computations of Segre classes of the cotangent bundle of a smooth complete intersection variety in an arbitrary ambient variety. Afterwards we give more details for a complete intersection in projective space.

In Chapter 3 we show that under the hypothesis of Debarre's conjecture all positivity conditions on the Chern classes that one might expect for an ample bundle are indeed satisfied by the Chern classes of the cotangent bundle of a complete intersection variety. More precisely we prove:

Theorem A. *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of sufficiently high multidegree and suppose that* $\mathrm{codim}_{\mathbb{P}^N}(X) \geq \dim(X)$*. Then Ω_X is numerically positive.*

The definition of numerically positive is motivated by a theorem of Fulton and Lazarsfeld [FL83]. We then give some explicit bounds on the multidegree in the case of surfaces. The proof of this result rests on the idea that, asymptotically, the Segre classes of the cotangent bundle somehow behave like the Chern classes of the normal bundle, which is known to be ample.

The content of Chapter 4 is a generalization to complete intersection varieties of a theorem of Diverio [Div09] on existence of jet differential equations on hypersurfaces. More precisely, we prove:

Theorem B. *Let M be a smooth N-dimensional projective variety and H an ample divisor on M. Fix $a \in \mathbb{N}$, fix $1 \leqslant c \leqslant N-1$ and $k \geqslant \lceil \frac{n}{c} \rceil$. Take $A_1, \ldots, A_c$ ample line bundles on M. For $d_1, \ldots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1 A_1|, \ldots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ is big on X_k. In particular when $m \gg 0$*

$$H^0(X, E_{k,m}\Omega_X \otimes H^{-ma}) \neq 0.$$

See section 1.4.3 for a definition of the varieties X_k. When looking at the case $M = \mathbb{P}^N$, we see that this result can be understood as a first positive argument towards Diverio-Trapani's conjecture. Moreover, thanks to a vanishing theorem of Diverio [Div08] we see that this result is optimal in k. The proof of Theorem B is based on the original idea of Diverio, that is to say the use of Demailly's holomorphic Morse inequalities. But we were also inspired by the work of Mourougane [Mou09] from which we borrowed, among other things, the systematic use of Segre classes instead of Chern classes.
Then we focus on the case $\kappa = 1$ and $M = \mathbb{P}^N$, the case of symmetric differential forms on complete intersections in projective space, for which we give an effective bound on the d_i. This case is of particular importance to us since this is the situation of Debarre's conjecture.

In Chapter 5 we use technics originally developed to solve Kobayashi's conjecture to give partial results towards Debarre's conjecture. These technics were first introduced by Siu [Siu04] and then detailed in [DMR10]. Replacing the study of holomorphic curves by the study of some "bad" curves we obtain:

Theorem C. *Let $X \subset \mathbb{P}^N$ be a generic complete intersection of sufficiently high multidegree and such that* $\mathrm{codim}_{\mathbb{P}^N}(X) \geqslant \dim(X)$*. Then Ω_X is ample modulo an algebraic subset of codimension at least two.*

As a corollary, we obtain a positive answer to Debarre's conjecture for surfaces.

Theorem D. *Let $N \geqslant 4$. Let $S \subset \mathbb{P}^N$ be a generic complete intersection surface of multidegree $(d_1, \ldots d_{N-2})$. If $d_i \geqslant \frac{8N+8}{N-3}$ for all $1 \leqslant i \leqslant N-2$, then Ω_S is ample.*

In particular, we obtain (non-explicit) examples of surfaces in $\mathbb{P}^4$ with ample cotangent bundle. Schneider already raised the question of the existence of such a surface in [Sch92] and, to our knowledge, it is the first known answer to Schneider's question.

In Chapter 6, we look at hyperbolicity of complete intersection varieties in an arbitrary algebraic variety. It is well known that a variety with ample cotangent bundle is hyperbolic, it is therefore natural to wonder whether under the hypothesis of Debarre's conjecture, the varieties we are considering are hyperbolic or not. Applying Diverio-Merker-Rousseau and Diverio-Trapani's results combined with a moving lemma we obtain:

Theorem E. *Let M be a smooth N-dimensional projective variety. Take $A_1, \ldots, A_c$ ample line bundles on M. For $d_1, \ldots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1 A_1|, \ldots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then, there exists an algebraic subset $Z \subset X$ of codimension at least $2c$ such that all the entire curves of X lie in Z. In particular, when $2c \geqslant \dim X$, then X is hyperbolic.*

In particular this contains the case $c \geqslant n$ and $\mathbb{P}^N$, which is the one considered in Debarre's conjecture. This result gives one more evidence towards this conjecture.

The idea of Chapter 7 is to use geometric constructions to obtain informations on the positivity of the cotangent bundle. First, as Frederic Han pointed out to us, it is interesting to study the "point-line" incidence variety. Doing so we obtain:

Theorem F. *Let $X \subseteq \mathbb{P}^N$. Then $\Omega_X(2)$ is ample if and only if X doesn't contain any line.*

Then we use, for computational purposes, the bundle $\widetilde{\Omega}_X$ which is roughly the bundle of differential forms on the cone over X that are invariant under the $\mathbb{C}^*$ action but do not necessarily satisfy Euler's condition. This bundle was in particular studied by Bogomolov and De Oliveira in [BDO08], but also by Debarre [Deb05]. This bundle is never ample but Debarre has proved that under the hypothesis of his conjecture, $\widetilde{\Omega}_X(1)$ is ample. It turns out that $\widetilde{\Omega}_X$ is somehow more suitable for computations than the cotangent bundle. Using this, we are able to compute explicitly some symmetric differential forms on some very particular complete intersection varieties. We are also able to obtain some generic vanishing results. We start by studying the cohomology of the bundles $S^k\widetilde{\Omega}_X(-a)$ on a complete intersection variety, this gives some vanishing results and allows us to develop a computation strategy. We then illustrate this strategy by constructing an example of a family of complete intersection surfaces with noteworthy properties.

Example. *Let $e \geqslant 5$, $e_1 := \lfloor \frac{e}{2} \rfloor$ and $e_2 := \lceil \frac{e}{2} \rceil$ (so that $e = e_1 + e_2$). For $\alpha := (\alpha_1, \alpha_2) \in \mathbb{C}^2$ and $\beta := (\beta_1, \beta_2) \in \mathbb{C}^2$ set,*

$$\begin{aligned} F_\alpha &:= X_0^e + X_1^e + X_2^e + X_3^e + X_4^e + \alpha_1 X_0^{e_1} X_1^{e_2} + \alpha_2 X_2^{e_1} X_3^{e_2}, \\ G_\beta &:= a_0 X_0^e + a_1 X_1^e + a_2 X_2^e + a_3 X_3^e + a_4 X_4^e + \beta_1 X_0^{e_1} X_1^{e_2} + \beta_2 X_2^{e_1} X_3^{e_2}. \end{aligned}$$

Set $S_\beta^\alpha := (F_\alpha = 0) \cap (G_\beta = 0) \subset \mathbb{P}^4$. For $(a_i)_{0 \leqslant i \leqslant 4} \in \mathbb{C}^5$ generic, this gives us a family of surfaces on $\mathbb{P}^4$. We have,

- $H^0(S_0^0, S^2\Omega_{S_0^0}) \neq 0$.
- *If $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{P}^4$ is generic, then $H^0(S_\beta^\alpha, S^2\Omega_{S_\beta^\alpha}) = 0$.*

Moreover we are able to describe explicitly all the members of $H^0(S_0^0, S^2\Omega_{S_0^0})$ and to give the equations of the base locus. This example shows that the numbers $h^0(X, S^m\Omega_X)$ fail to be deformation invariant. This phenomenon was already observed in a more general setting in [Bog78] and [BDO08]. The work we present in this chapter is still unfinished, however we hope that it might already be of some interest.

In Chapter 8 we generalize a vanishing theorem due to Schneider [Sch92] for symmetric powers of the cotangent bundle of a projective variety. Our result is the following.

Theorem G. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geqslant 1$, k integers $\ell_1, \dots, \ell_k \geqslant 0$ and $a \in \mathbb{Z}$.*

1. *If $\ell_1 + \cdots + \ell_k > a + k$ and $j < n - k \cdot c$ then*
$$H^j(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$
2. *If $\ell_1 + \cdots + \ell_k > a$ and $0 < n - k \cdot c$ then*
$$H^0(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Schneider proved the case $k = 1$ of this theorem. Our proof of Theorem G uses roughly the same idea than the one present in [Sch92], which is that one can use the positivity of the normal bundle of a subvariety of a projective space combined with some deep vanishing results. However, were Schneider used LePotier's vanishing theorem, we will need a more general statement due to Ein-Lazarsfeld [EL93] and Manivel [Man97]. From this statement, one can deduce a new vanishing result for Green-Griffiths jet differentials.

Theorem H. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension c. Let $a \in \mathbb{Z}$. If $1 \leqslant k < \frac{n}{c}$ and $a < \frac{m}{k}$ then*

$$H^0(X, E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

This result therefore shows that the existence of Green-Griffiths jet differentials imposes embedding restrictions which are higher order analogues to the ones pointed out in [Sch92]. Diverio [Div08] already proved a theorem in that direction. Diverio's result is exactly Theorem H under the assumption that X is a complete intersection variety and $a = 0$.
Pacienza and Rousseau used the same idea as Diverio to prove a vanishing result for their generalized Green-Griffiths jet differential bundles on complete intersection varieties, therefore it is not surprising that using Theorem G one can drop the complete intersection assumption. Moreover one can also give a similar result for higher order cohomology groups, this is the content of Corollary 8.4.2.

In chapter 9 we tackle a different problem which arose naturally while we were studying the cohomology groups of the symmetric powers of the cotangent bundle of a complete intersection variety. The question is to determine the cohomology groups of line bundles on universal hypersurfaces. This turns out to be a non-trivial question and we were only able to solve it in particular cases. Our result is the following.

Theorem I. *Let $\mathcal{H} := \{(t,x) \in \mathbb{P}^{N_d} \times \mathbb{P}^1 \ / \ t(x) = 0\}$, where $\mathbb{P}^{N_d} := \mathbb{P}\left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^*\right)$. Then, $\forall\ m_1, m_2 \in \mathbb{Z}$ all the morphisms appearing in the long exact sequence associated to the restriction short exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^{N_d} \times \mathbb{P}^1}(m_1 - 1, m_2 - d) \to \mathcal{O}_{\mathbb{P}^{N_d} \times \mathbb{P}^1}(m_1, m_2) \to \mathcal{O}_{\mathcal{H}}(m_1, m_2) \to 0$$

are of maximal rank.

This allows us to obtain many informations on the cohomology groups of $\mathcal{O}_{\mathcal{H}}(m_1, m_2)$. We believe that this result should also hold in greater dimensions but, up to now, we were unable to deal with the combinatorics involved. The proof is just an intricate Gauss algorithm.

Notation and conventions

Let us present the notation we are using in this thesis. We believe that it is all standard.

- We will work over the field of complex numbers $\mathbb{C}$.
- $\mathbb{P}^N$ denotes the N-dimensional projective space.
- Usually, X will denote an n-dimensional smooth connected projective variety.
- TX is the tangent bundle of X.
- $\Omega_X = TX^*$ is the cotangent bundle of X.
- $K_X = \det(\Omega_X)$ the canonical bundle.
- $E^{GG}_{k,m}\Omega_X$ is the Green-Griffiths jet differential bundle of order k and degree m (see section 1.4.2).
- $E_{k,m}\Omega_X$ is the Demailly-Semple jet differential bundle of order k and degree m (see section 1.4.3).
- In what follows E is a vector bundle over X.
- $\mathbb{P}(E) \to X$ denotes the projectivized bundle of *rank-one quotients* of E, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the tautological line bundle on $\mathbb{P}(E)$.
- $P(E) \to X$ denotes the projectivized bundle of *lines* in E, and $\mathcal{O}_{P(E)}(-1)$ is the tautological line bundle on $P(E)$.

- $c_i(E)$ denotes the i-th Chern class of E.
- $s_i(E)$ denotes the i-th Segre class of E.
- Let L be a line bundle on X, we define the stable base locus of L by

  $$\mathrm{Bs}(L) := \bigcap_{\substack{m>0 \\ s\in H^0(X,mL)}} (s=0),$$

  where $(s=0)$ is the zero locus of the section s.
- If L is an ample line bundle on X, we will say that a property P holds for a *generic* member of $|L|$, if there exists a non-empty Zariski open subset $U \subseteq |L|$ such that the property P holds for each element in U.
- For $N > 0$ and $d \geqslant 0$ we set $\mathbb{P}^{N_d} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}))$. To each element $t = (t_1, \ldots, t_c) \in \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_c}$ corresponds the variety $X_t \subset \mathbb{P}^N$ whose equations are $t_1 = 0, \ldots, t_c = 0$. We will say that a property P holds for a *generic* complete intersection in $\mathbb{P}^N$, of codimension c and multidegree $(d_1, \ldots, d_c)$, if there exists a non-empty Zariski open subset $U \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_c}$ such that given any $t \in U$, P holds for X_t.

Résumé en français

Une des façons de comprendre la géométrie d'une variété algébrique est de comprendre les courbes qu'elle contient. En particulier, l'étude des courbes rationnelles joue un rôle clé dans la classification birationnelle des variétés. De façon plus générale, en quittant le monde purement algébrique, on peut chercher à comprendre les courbes entières dans une variété complexe. Pour fixer les idées, prenons une variété projective lisse X. Une *courbe entière* de X est une application entière $f : \mathbb{C} \to X$ non constante. Dans cette direction, plusieurs questions se posent naturellement.

1. Existe-t-il des courbes entières dans X ?
2. Existe-t-il un sous-ensemble algébrique strict de X contenant toutes les courbes entières ?
3. Pour chaque courbe entière, existe-t-il un sous-ensemble algébrique strict de X la contenant ?

Un instant de réflexion nous convainc qu'une réponse négative à la première question entraîne une réponse positive à la seconde, qui à son tour entraîne une réponse positive à la troisième question. Nous dirons que X est *hyperbolique* si elle ne contient pas de courbe entière.

Il est en général très difficile de déterminer si une variété est hyperbolique ou pas. Plusieurs conjectures majeures guident ce type de recherches ; nous en citons deux.

Conjecture (Kobayashi)**.** *Une hypersurface générique de grand degré dans un espace projectif est hyperbolique.*

Conjecture (Green-Griffiths)**.** *Une variété de type général contient un sous-ensemble algébrique strict qui contient toutes les courbes entières non constantes.*

La conjecture de Green-Griffiths est particulièrement profonde car elle prédit qu'une condition de positivité sur le fibré canonique (qui est une condition de nature purement algébrique) entraînerait des conséquences très fortes sur le comportement des courbes entières (qui sont typiquement de nature purement transcendante). Cette conjecture est encore largement ouverte.

Cette thèse vise à étudier ce genre de problématiques pour les variétés intersection complète. Dans notre travail, nous utiliserons l'approche des différentielles de jets, mais il faut noter qu'il y a d'autres approches. Par exemple, nous mentionnons l'existence de techniques très importantes provenant de la théorie de Nevanlinna et de la théorie des feuilletages.

Il est possible d'utiliser des outils de géométrie algébrique pour étudier les courbes entières contenues dans une variété projective. Les idées de la théorie des différentielles de jets remontent aux travaux de Bloch [Blo26] mais cette théorie a surtout été développée dans les travaux de Green et Griffiths. Dans [GG80], à chaque variété X ils associent ce qui est maintenant appelé *les fibrés de différentielles de jets de Green-Griffiths* $E^{GG}_{k,m}\Omega_X$ qui sont, *grosso modo*, des faisceaux d'équations différentielles d'ordre k et de degré m. Plus précisément, en tout point $x \in X$ la fibre $E^{GG}_{k,m}\Omega_{X,x}$ est exactement constituée des polynômes de la forme

$$P\left(f', \dots, \left(f^{(k)}\right)\right) = \sum_{\substack{I_1,\dots,I_k \in \mathbb{N}^n \\ |I_1|+2|I_2|+\cdots+k|I_k|=m}} a_{I_1,\dots,I_k} (f')^{I_1} \cdots (f^{(k)})^{I_k},$$

où $n = \dim X$ et où $\left(f', \cdots, f^{(k)}\right)$ est le k-jet d'un germe de courbe holomorphe $f : (\mathbb{C}, 0) \to (X, x)$. Pour que cette expression ait un sens, il faudrait prendre des coordonnées locales autour de x, écrire $f = (f_1, \dots, f_n)$ et ensuite utiliser la notation standard pour les multi-indices ; pour $I = (i_1, \dots, i_n)$ et $1 \leqslant j \leqslant k$ on définit $(f^{(j)})^I := (f_1^{(j)})^{i_1} \cdots (f_n^{(j)})^{i_n}$. L'importance de ces fibrés réside dans leurs applications aux problèmes d'hyperbolicité via le critère d'annulation. Le critère d'annulation s'écrit : *toute équation différentielle de jets globale s'annulant sur un diviseur ample doit être vérifiée par toutes les courbes entières.*

Plus tard, Demailly [Dem97a] a introduit une version raffinée des ces fibrés, appelés maintenant *les fibrés de différentielles de jets de Demailly-Semple* ou *fibrés de differentielles de jets invariants* $E_{k,m}\Omega_X$. Son idée était que si l'on étudie les courbes entières tracées dans X, alors ce qui nous intéresse est seulement le lieu géométrique des courbes en question et non la façon dont elles sont paramétrées. Demailly a alors considéré le sous-fibré $E_{k,m}\Omega_X$ de $E_{k,m}^{GG}\Omega_X$ constitué des éléments invariants par reparamétrage des jets de courbes. Clairement, le critère d'annulation est toujours valide dans ce contexte et, en vue de celui-ci, le problème de déterminer des éléments non nuls de $H^0(X, E_{k,m}^{GG}\Omega_X \otimes A^{-1})$ (ou $H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$) pour un certain fibré en droites ample A est devenu un problème central.

Pour illustrer cela, l'approche des différentielles de jets invariants a récemment permis de montrer qu'une hypersurface générique de grand degré dans l'espace projectif contient un sous-ensemble algébrique qui contient l'image de toutes les courbes entières. Ce travail a été fait par Diverio, Merker et Rousseau [DMR10] et est basé sur une stratégie de Siu [Siu04].

Un cas particulier de cette technique concerne la positivité du fibré cotangent. Plus précisément, prenons une variété projective X. Si le fibré cotangent de X est ample alors X est hyperbolique. Ceci dit, l'amplitude du fibré cotangent est une hypothèse extrêmement forte et très difficile à vérifier. Les variétés à fibré cotangent ample sont supposées être relativement abondantes et il est notable qu'il n'y ait que peu d'exemples et de constructions connus. La construction de tels exemples est le problème principal qui nous occupe durant cette thèse, motivé par la conjecture de Debarre [Deb05] suivante.

Conjecture (Debarre)**.** *Si $X \subset \mathbb{P}^N$ est une variété intersection complète générique de multidegré suffisamment grand telle que* $\operatorname{codim}_{\mathbb{P}^N}(X) \geqslant \dim(X)$*, alors le fibré cotangent de X est ample.*

Nous donnerons différents résultats partiels en direction de cette conjecture, et en particulier, nous parvenons à démontrer cette conjecture dans le cas des surfaces. D'autre part, il est intéressant de comprendre les cas de petite codimension. Ceci peut être abordé d'un point de vue cohomologique, ce qui est l'approche suggérée par Debarre. L'autre approche provient des différentielles de jets. Diverio et Trapani [DT09] proposent une conjecture précise qui est une sorte d'interpolation entre la conjecture de Kobayashi et la conjecture de Debarre. Nous donnerons un premier résultat en direction de cette conjecture.

Conjecture (Diverio-Trapani)**.** *Si $X \subset \mathbb{P}^N$ est l'intersection d'au moins $\frac{N}{k+1}$ hypersurfaces génériques de grand degré, alors $E_{k,m}\Omega_X$ est ample pour tout $m > 0$.*

Contenu de la thèse

Nous décrivons maintenant le contenu détaillé de la thèse et nous donnons les principales idées et les principaux éléments de démonstration. Nous travaillons sur le corps des nombres complexes $\mathbb{C}$. Si rien n'est précisé, X désignera une variété algébrique complexe projective lisse de dimension n.

Chapitre 1

Dans le chapitre 1 nous commençons par rappeler les principales notions de positivité pour les fibrés en droites et le fibrés vectoriels dont nous avons besoin. Les deux principales notions de positivité sont la notion de fibré *ample* et la notion de fibré *gros*.

Définition 1. Soit L un fibré en droites sur X. Alors L est dit :

- *très ample* s'il existe un plongement $\iota : X \to \mathbb{P}^N$ tel que $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq L$.
- *ample* s'il existe $m > 0$ tel que mL est très ample.
- *très gros* s'il existe une application rationnelle $\iota : X \dashrightarrow \mathbb{P}^N$ birationnelle sur son image telle que $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq L$.
- *gros* s'il existe $m > 0$ tel que mL est très gros.

Les notions d'*ample* et *gros* sont des notions « ouvertes ». En effet dans le groupe de Néron-Severi $N^1(X)$, le cône engendré par les classes de diviseurs amples $\mathrm{Amp}(X)$, et le cône engendré par les classes de diviseurs gros $\mathrm{Big}(X)$, sont tous deux des cônes ouverts. On définit alors $\mathrm{Nef}(X) := \overline{\mathrm{Amp}(X)}$ et $\mathrm{Pseff}(X) := \overline{\mathrm{Big}(X)}$. On dira alors qu'un diviseur est *nef* si sa classe dans $N^1(X)$ appartient à $\mathrm{Nef}(X)$, et qu'un diviseur est *pseudoeffectif* si sa classe appartient à $\mathrm{Pseff}(X)$. Nous rappelons dans la section 1.1.2 quelques résultats essentiels concernant ces différentes notions de positivité pour les fibrés en droites.

Dans la section 1.2 nous expliquons comment étendre ces notions aux fibrés vectoriels. Le procédé standard consiste à considérer le fibré en droites tautologique sur le projectivisé du fibré vectoriel en question. Plus précisément, soit $E \to X$ un fibré vectoriel sur X. On note $\mathbb{P}(E) \overset{\pi_E}{\to} X$ le projectivisé des quotients de rang 1 de E (notation de Grothendieck). Sur la variété $\mathbb{P}(E)$ on a naturellement un fibré en droites $\mathcal{O}_{\mathbb{P}(E)}(1)$, dit tautologique. On dira alors que E est *ample* (resp. *nef*) si $\mathcal{O}_{\mathbb{P}(E)}(1)$ est ample (resp. nef). On pourrait bien évidemment faire de même pour *gros*, mais il nous semble important de signaler une subtilité ici : le problème vient du lieu base stable du fibré $\mathcal{O}_{\mathbb{P}(E)}(1)$. Il se pourrait très bien que le lieu base stable se surjecte sur X sous la projection π_E. Il est donc naturel de distinguer le cas où le lieu base stable se surjecte sur X du cas où le lieu base stable ne se surjecte pas sur X.

Définition 2. Soit E un fibré vectoriel sur X. Alors E est dit :

- *faiblement gros* si $\mathcal{O}_{\mathbb{P}(E)}(1)$ est gros sur $\mathbb{P}(E)$.
- *fortement gros* si $\mathcal{O}_{\mathbb{P}(E)}(1)$ est gros sur $\mathbb{P}(E)$ et si de plus $\pi_E(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subsetneq X$.

Nous introduisons aussi la notion de *positivité numérique*. Pour un fibré en droites L sur X, on peut tester l'amplitude en étudiant la positivité de la première classe de Chern $c_1(L)$. Plus précisément, L est ample si et seulement si pour toute sous-variété $V \subseteq X$, $\int_V (c_1(L))^{\dim V} > 0$. Mais, on ne peut pas espérer un résultat analogue pour les fibrés vectoriels de rang supérieur, à part dans des cas très particuliers. Cependant, Fulton et Lazarsfeld ont explicité précisément l'ensemble des polynômes en les classes de Chern qui doivent être positifs pour tous les fibrés vectoriels amples. Si λ est une partition de ℓ alors on note $\Delta_\lambda(c(E))$, le polynôme de Schur en les classes de Chern de E associé à la partition λ (voir section 1.2.2). Le théorème de Fulton et Lazarsfeld stipule que si E est ample, si $Y \subseteq X$ est une sous-variété de dimension ℓ et si λ est une partition de ℓ alors $\int_Y \Delta_\lambda(c(E)) > 0$. En général il n'y a pas équivalence. Nous dirons qu'un fibré vectoriel est *numériquement positif* s'il vérifie les conclusions du théorème de Fulton et Lazarsfeld.

La dernière notion qui nous intéressera est la notion de fibré *ample presque partout* due à Miyaoka. L'idée vient de l'observation suivante : pour montrer qu'un fibré vectoriel E est ample sur X, il suffit de montrer que pour un diviseur ample H sur X, $E \otimes H^{-1}$ est nef. Par définition, cela revient à montrer que $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi_E^* H^{-1}$ est nef sur $\mathbb{P}(E)$. Le critère de Kleimann nous dit qu'il est suffisant de montrer que pour toute courbe irréductible $C \subseteq \mathbb{P}(E)$, $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \otimes \pi_E^* H^{-1} \geq 0$, ou de façon équivalente, $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)(1)}) \geq C \cdot \pi_E^* H$. Il est donc intéressant de localiser les courbes $C \subseteq \mathbb{P}(E)$ telles que $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)(1)}) < C \cdot \pi_E^* H$, en effet, s'il n'y a pas de telle courbe alors E est ample. On a la définition suivante due à Miyaoka.

Définition 3. Soit E un fibré vectoriel sur X. Soit H un fibré en droites ample sur X et soit T un sous-ensemble de X. Nous dirons que E est *ample modulo* T si pour $\epsilon > 0$ suffisamment petit, toute courbe irréductible $C \subseteq \mathbb{P}(E)$ telle que $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)(1)}) < \epsilon C \cdot \pi_E^* H$ vérifie $\pi_E(C) \subseteq T$. Nous dirons que E est *ample presque partout* si E est ample modulo un sous-ensemble algébrique strict de X.

Après avoir défini toutes ces notions de positivité nous passons à une courte introduction sur l'hyperbolicité au sens de Kobayashi.

Ensuite nous rappelons quelques propriétés des variétés dont le fibré cotangent est ample (section 1.3.2). Une des propriétés remarquables est qu'une variété à fibré cotangent ample est hyperbolique. Déterminer si une variété est hyperbolique ou non, ou si une variété a un fibré cotangent ample sont, en général, des questions très difficiles. Et il n'y a que relativement peu d'exemples connus de variété à fibré cotangent ample. Nous rappelons alors quelques exemples. Miyaoka, se basant sur des idées de Bogomolov, a construit des exemples de surfaces à fibré cotangent ample comme intersection complète dans un produit de deux surfaces à fibré cotangent ample presque partout.
Bogomolov a construit des variétés à fibré cotangent ample comme intersection complète dans un produit de variété à fibré cotangent faiblement gros (construction qui a été détaillée par Debarre dans [Deb05]).
Debarre a construit des variétés à fibré cotangent ample comme intersection complète dans une variété abélienne. Motivé par ce résultat, Debarre a conjecturé le résultat analogue dans l'espace projectif. Cette conjecture est le point de départ de notre travail.

Nous passons ensuite à une introduction succincte sur les équations différentielles de jets, et sur leurs applications aux problèmes d'hyperbolicité. Des détails concernant l'application des techniques de différentielles de jets au problème de la dégénérescence des courbes algébriques dans les hypersurfaces génériques de $\mathbb{P}^N$ se trouvent dans la section 1.4.

La dernière partie de ce chapitre est consacrée au liens entre l'hyperbolicité et l'existence de points rationnels sur X quand X est définie sur un corps de nombre ou un corps de fonctions. Ce domaine est encore très largement conjectural. Les conjectures de Lang y jouent un rôle central.

Conjecture 1 (Lang)**.** *Soit X une variété définie sur un corps de nombres K. Alors X est hyperbolique si et seulement si pour toute extension finie L de K, X ne contient qu'un nombre fini de L-points.*

Comme les variétés à fibré cotangent ample sont hyperboliques, on obtient une autre conjecture plus faible mais encore largement ouverte.

Conjecture 2 (Lang)**.** *Soit X une variété définie sur un corps de nombre K. Si le fibré cotangent de X est ample alors X ne contient qu'un nombre fini de K-points.*

Si X est une courbe, ces conjectures se réduisent au problème de Mordell, qui a été démontré par Faltings. La conjecture 2 a été démontrée par Moriwaki sous l'hypothèse supplémentaire que le fibré cotangent est globalement engendré. L'analogue de la conjecture 2 sur les corps de fonctions a été démontré par Noguchi.

Chapitre 2

Le chapitre 2 est consacré aux calculs d'intersections sur les variétés intersection complète. Ces calculs sont à la base de plusieurs de nos résultats ; c'est pour cela qu'ils sont regroupés dans un chapitre à part. Il est à noter que nous utilisons systématiquement les classes de Segre à la place des classes de Chern. Bien entendu, d'un point de vue théorique ces deux approches sont les mêmes, mais l'utilisation des classes de Segre apporte des simplifications considérables en pratique. On commence ce chapitre en regardant comment les calculs d'intersection dans la tour des jets de Demailly s'effectuent en fonction de l'intersection des classes de Segre sur la variété de base. Puis nous détaillons les calculs d'intersection des classes de Segre sur une variété intersection complète lisse dans une variété projective lisse ambiante quelconque. Sur une variété intersection complète X dans une variété M, on considère les espaces de jets projectivisés de Demailly X_k munis de leur fibré tautologique $\mathcal{O}_{X_k}(1)$ (voir section 1.4.3). On note alors $u_k := c_1(\mathcal{O}_{X_k}(1))$, $s_{k,i} := s_i(\mathcal{F}_k)$, $s_i := s_i(\Omega_X)$ et

$h := c_1(H)$ où H est un diviseur ample fixé. On note aussi $\mathcal{C}_k(X,H) := \mathbb{Z}\cdot u_k \oplus \cdots \oplus \mathbb{Z}\cdot u_1 \oplus \mathbb{Z}\cdot h \subset N^1(X_k)$. Le problème principal est d'estimer des calculs d'intersection impliquant les u_k. L'idée principale est de « pousser » les calculs d'intersection sur X où l'on peut naturellement les exprimer en terme des classes de Segre du fibré cotangent.

Lemme 1. *Soit $k \geqslant 0$, $a \geqslant 0$, $\ell \geqslant 0$. Prenons ℓ entiers positifs $i_1, \ldots, i_\ell$ et m classes de diviseurs $\gamma_1, \ldots, \gamma_m \in \mathcal{C}_k(X,H)$ tels que $i_1 + \cdots + i_\ell + m + a = n_k$. Soit $\gamma_q =: \alpha_{q,0}h + \sum_i \alpha_{q,i}u_i$. Alors*

$$\int_{X_k} s_{k,i_1} \cdots s_{k,i_\ell} \gamma_1 \cdots \gamma_m h^a = \sum_{j_1,\ldots,j_{k+\ell},b} Q_{j_1,\ldots,j_{k+\ell},b} \int_X s_{j_1} \cdots s_{j_{k+\ell}} h^{a+b},$$

où dans chaque terme de la somme on a $b \geqslant 0$ et où les $Q_{j_1,\ldots,j_{k+\ell},b}$ sont des polynômes en les $\alpha_{q,i}$ dont les coefficients sont indépendants de X. De plus, quitte à réordonner les j_p on a $j_1 \leqslant i_1, \ldots, j_\ell \leqslant i_\ell$.

Il reste alors à estimer l'intersection des classes de Segre du fibré cotangent. Pour fixer les notations, on suppose que X est l'intersection d'éléments de $|d_1A_1|, \ldots, |d_cA_c|$ où les A_i sont des diviseurs amples sur M. Les nombre d'intersections que l'on calcul sont alors des polynômes en d_i, et on peut obtenir des estimations sur le degré de ces polynômes.

Lemme 2. a) *Soit $0 \leqslant i_1 \leqslant \cdots \leqslant i_k$, $\ell > 0$ tels que $i_1 + \cdots + i_k + \ell = n$. Alors*

$$\deg\left(\int_X s_{i_1} \cdots s_{i_k} h^\ell\right) < N.$$

b) *Soient $0 \leqslant i_1 \leqslant \cdots \leqslant i_k$ tels que $i_1 + \cdots + i_k = n$. Alors $\int_X s_{i_1} \cdots s_{i_k}$ est de degré N si et seulement si $i_k \leqslant c$.*

c) *Soient $0 \leqslant i_1 \leqslant \cdots \leqslant i_\kappa$. Si $i_1 < b$ ou si $i_1 = b$ et $i_j < c$ pour un certain $j > 1$, alors*

$$\deg\left(\int_X s_{i_1} \cdots s_{i_\kappa}\right) < N.$$

Nous détaillons encore plus ces calculs dans le cas où $M = \mathbb{P}^N$, ce qui nous permettra d'obtenir des résultats effectifs par la suite.

Chapitre 3

Pour tester si la conjecture de Debarre peut être vraie, une des premières étapes est de vérifier que le fibré cotangent d'une variété intersection complète générique de grand multidegré et de grande codimension dans $\mathbb{P}^N$ est numériquement positif. Dans le chapitre 3 nous nous attaquons à ce problème. La combinatoire initiale de ce problème paraît assez compliquée car elle fait intervenir toute la combinatoire des polynômes de Schur en plus de faire intervenir la combinatoire des classes de Segre du fibré cotangent. Cependant, une petite astuce nous permet de passer outre ces calculs en appliquant le théorème de Fulton et Lazarsfeld au fibré normal de X dans $\mathbb{P}^N$. L'idée se trouve dans la suite exacte du fibré conormal

$$0 \to \bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i) \to \Omega_{\mathbb{P}^N|_X} \to \Omega_X \to 0,$$

où c est la codimension de X dans $\mathbb{P}^N$ et où $(d_1, \ldots, d_c)$ est le multidegré de X. En appliquant la formule de Whitney on observe qu'asymptotiquement (en d_i) les classes de Segre de Ω_X sont proches des classes de Chern de $\bigoplus_{i=1}^c \mathcal{O}_X(d_i)$. Un petit calcul nous donne alors le résultat suivant.

Théorème A. *Soit $a \in \mathbb{Z}$. Il existe $D_{N,n,a} \in \mathbb{N}$ tel que si $X \subset \mathbb{P}^N$ est une intersection complète lisse de dimension n, de codimension c et de multidegré $(d_1, \ldots, d_c)$ telle que $c \geqslant n$ et $d_i > D_{N,n,a}$ pour tout i, alors $\Omega_X(-a)$ est numériquement positif.*

Si l'on cherche à avoir une borne effective pour $D_{N,n,a}$ il faut faire les calculs explicitement, ce qui en général est très complexe. Par contre dans le cas des surfaces il est possible de faire tous les calculs sans trop de problèmes. On trouve $D_{N,N-2,0} = \delta(N)$ où $\delta(4) = 9$, $\delta(5) = 5$, $\delta(N) = 4$ pour $N = 6, 7$, $\delta(N) = 3$ pour$8 \leqslant N \leqslant 12$, $\delta(N) = 2$ pour $13 \leqslant N$.

Chapitre 4

Dans le chapitre 4 nous généralisons un résultat de Diverio. L'objectif de ce chapitre est de démontrer le théorème suivant.

Théorème B. *Soient M une variété projective lisse de dimension N et H un fibré en droites ample sur$M$. Soit $a \in \mathbb{N}$, soient $1 \leqslant c \leqslant N-1$ et $k \geqslant \lceil \frac{n}{c} \rceil$. Prenons $A_1, \ldots, A_c$ des fibrés en droites amples sur M. Pour $d_1, \ldots, d_c \in \mathbb{N}$ suffisamment grands, prenons des éléments génériques $H_1 \in |d_1 A_1|, \ldots, H_c \in |d_c A_c|$ et posons $X = H_1 \cap \cdots \cap H_c$. Alors $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ est gros sur X_k. En particulier quand $m \gg 0$,*

$$H^0(X, E_{k,m}\Omega_X \otimes H^{-ma}) \neq 0.$$

Nous renvoyons à la section 1.4.3 pour la construction de variétés X_k. Quand $M = \mathbb{P}^N$, ce résultat peut être considéré comme un premier argument en faveur de la conjecture de Diverio et Trapani. Par un théorème d'annulation de Diverio [Div08], nous voyons que ce résultat est optimal en k. Dans le cas particulier où $M = \mathbb{P}^N$, $c = 1$, et $a = 0$ ce théorème a été démontré par Diverio [Div09]. La strategie de la preuve est sensiblement la même que la sienne, c'est-à-dire l'utilisation des inégalités de Morse holomorphes. Cependant nos calculs sont simplifiés par l'utilisation systématique des classes de Segre à la place des classes de Chern dans les calculs. La stratégie est la suivante. Diverio a observé que pour montrer que $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ est gros il suffit de montrer que $\mathcal{O}_{X_k}(a_1, \ldots, a_k) \otimes \pi_k^* H^{-a}$ est gros pour certains a_i bien choisis ($a_i > 0$ est suffisant). On introduit alors

$$L_k := \mathcal{O}_{X_k}(2 \cdot 3^{k-2}, \ldots, 2 \cdot 3, 2, 1) \otimes \pi_k^* H^{r_0 \cdot 3^{k-1}}.$$

On peut montrer que L_k est nef si $\Omega_M \otimes H^{r_0}$ est nef. Et on pose $F := L_k \otimes \cdots \otimes L_1$, et $G = \pi_k^* H^{m+a}$ où $m \geqslant 0$ est choisi de sorte que $F \otimes \pi_k^* H^{-m}$ ne contient pas d'éléments provenant de X. D'après l'observation de Diverio, il est suffisant de montrer que $F - G$ est gros. Pour ce faire, nous utiliserons un cas particulier des inégalités de Morse holomorphes de Demailly.

Théorème 1 (Inégalités de Morse holomorphes)**.** *Soit Y une variété projective lisse de dimension n et soient F_0 et G_0 des diviseurs nef sur Y. Si $F_0^n > nG_0 \cdot F_0^{n-1}$, alors $F_0 - G_0$ est gros.*

Au vu de ce théorème, il nous suffit alors de montrer que,

$$F^{n_k} > n_k F^{n_k - 1} \cdot G,$$

où $n_k = \dim(X_k)$. L'idée est maintenant de montrer que F^{n_k} et $n_k F^{n_k-1} \cdot G$ sont tous deux des polynômes en d_i, et de montrer que F^{n_k} est plus grand qu'un polynôme positif de degré N alors que $n_k F^{n_k-1} \cdot G$ est un polynôme de degré strictement inférieur à N. Pour démontrer cela on s'appuie fortement sur les calculs du chapitre 2 dont on déduit le lemme technique suivant. On note

$$\ell_k := c_1(L_k) = u_k + \beta_k,$$

où β_k est la classe d'un diviseur qui provient de X_{k-1}.

Lemme 3. *Avec les mêmes notations que précédemment.*

a) *Soit $k \geqslant 1$ et soient $\gamma_1, \ldots, \gamma_{n_k - 1} \in \mathcal{C}_k(X, H)$. Alors*

$$\int_{X_k} \gamma_1 \cdots \gamma_m h = o(d^N).$$

b) *Soient* $\gamma_1, \ldots, \gamma_p \in \mathcal{C}_k(X,H)$ *et* $0 \leqslant i_1 \leqslant \cdots \leqslant i_q$ *tels que* $p + \sum i_j = n_k$. *Si* $i_1 < b$, *ou si* $i_1 = b$ *et* $i_j < c$ *pour un certain* $j > 1$, *alors*

$$\begin{aligned} \int_{X_k} s_{k,i_1} \cdots s_{k,i_q} \gamma_1 \cdots \gamma_p &= o(d^N), \\ \int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q} \gamma_1 \cdots \gamma_p &= o(d^N). \end{aligned}$$

c) *Soit* $0 < k < \kappa$. *Alors*

$$\int_{X_k} s_{k,b} s_{k,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} = \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-k} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N).$$

Ce lemme nous permettra à la fois de montrer que $n_k F^{n_k - 1} \cdot G$ est un polynôme de degré strictement inférieur à N et de montrer que

$$F^{n_k} \geqslant \int_X \left(s_b s_c^{k-1}\right)^{dom} + o(d^N).$$

Et il suffit de conclure en montrant que $\int_X \left(s_b s_c^{k-1}\right)^{dom}$ est un polynôme positif de degré N.

Ensuite, nous nous intéressons au cas $\kappa = 1$ (c'est-à-dire le cas des formes différentielles symétriques), qui est d'une importance particulière car il s'inscrit dans le cadre de la conjecture de Debarre. Nous donnons une borne effective pour les d_i dans ce cas.

Chapitre 5

Le chapitre 5 est le point d'orgue du lien entre la conjecture de Debarre et la conjecture de Kobayashi. En effet, nous adaptons des techniques développées en vue de résoudre la conjecture de Kobayashi pour donner des résultats en direction de la conjecture de Debarre. Ces techniques trouvent leurs origines dans les travaux de Siu [Siu04], et ont été détaillées par la suite dans les travaux de Diverio, Merker et Rousseau [DMR10]. En remplaçant l'étude des courbes entières par l'étude de certaines « mauvaises » courbes nous obtenons :

Théorème C. *Soit* $X \subset \mathbb{P}^N$ *une variété intersection complète générique de suffisamment grand multidegré et telle que* $\operatorname{codim}_{\mathbb{P}^N}(X) \geqslant \dim(X)$. *Alors* Ω_X *est ample modulo un lieu de codimension au moins deux.*

En particulier, nous obtenons une réponse positive à la conjecture de Debarre pour les surfaces.

Théorème D. *Soit* $N \geqslant 4$. *Soit* $S \subset \mathbb{P}^N$ *une surface intersection complète générique de multidegré* $(d_1, \cdots d_{N-2})$. *Si pour tout* i, $d_i \geqslant \frac{8N+8}{N-3}$, *alors* Ω_S *est ample.*

La preuve se base sur les mêmes idées que la récente preuve de la dégénérescence algébrique des courbes entières dans une hypersurface algébrique générique de grand degré dans $\mathbb{P}^N$ donnée par Diverio Merker et Rousseau [DMR10] basée sur une stratégie de Siu. Le premier ingrédient est la construction de formes différentielles symétriques sur les intersections complètes lisses de grand multidegré et de grande codimension, ceci a été fait au chapitre 4. L'autre ingrédient est l'existence de certains champs de vecteurs méromorphes sur le projectivisé du fibré cotangent relatif sur l'intersection complète universelle. Plus précisément : soit $\mathbf{P} := \mathbb{P}^{N_{d_1}} \times \cdots \times \mathbb{P}^{N_{d_c}}$, où $\mathbb{P}^{N_{d_i}} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i))^*)$ est l'espace des paramètres, et soit $\mathcal{X} := \{(x, (t_1, \ldots, t_c)) \in \mathbb{P}^N \times \mathbf{P} \;\;/\;\; t_i(x) = 0 \;\;\forall i\}$ l'intersection complète universelle. Nous notons $\rho_1 : \mathcal{X} \to \mathbb{P}^N$ la projection sur le premier facteur et $\rho_2 : \mathcal{X} \to \mathbf{P}$ celle sur le second facteur. On utilise la notation standard $\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_c)$ pour les fibré en droites sur $\mathbf{P}$. De plus, on note $\pi : \mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \to \mathcal{X}$ la projection standard. Avec ces notations, le deuxième résultat préliminaire est le suivant.

Théorème 2. *Le fibré*

$$T\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}}) \otimes \pi^* \rho_1^* \mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^* \rho_2^* \mathcal{O}_{\boldsymbol{P}}(1, \ldots, 1)$$

est globalement engendré sur $\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}})$.

La preuve de ce théorème se déduit de la preuve par Merker du résultat principal de son article [Mer09]. Nous donnons quelques éléments de démonstration dans la section 5.2.

La preuve du théorème C consiste à remarquer dans un premier temps que le problème se réduit à montrer que le lieu base stable de $\mathcal{O}_{\mathbb{P}(\Omega_X)} \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-1)$ ne se surjecte pas sur X. Il faut alors estimer ce lieu base. Il s'agit alors de construire un élément non nul

$$\sigma \in H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)\right),$$

pour un certain ouvert de Zariski U. Ceci peut être fait grâce aux résultats du chapitre 4. Et ensuite, de différentier cette section par rapport à différents champs de vecteurs méromorphes bien choisis que l'on construit grâce au théorème 2. Nous construisons ainsi beaucoup de nouveaux éléments de

$$H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka)\right).$$

Il suffit alors de restreindre toutes ces sections à une fibre $\mathbb{P}(\Omega_{X_t})$ avec $t \in U$. On peut alors montrer par un calcul local que les sections ainsi construites découpent un lieu base qui ne se surjecte pas sur X_t.

Chapitre 6

Dans le chapitre 6, nous revenons sur des problèmes d'hyperbolicité. Nous savons qu'une variété à fibré cotangent ample est hyperbolique ; il est donc intéressant de savoir si, sous les hypothèses de la conjecture de Debarre, les variétés intersection complète que l'on considère sont hyperboliques. L'idée consiste simplement à utiliser le théorème de Diverio, Merker et Rousseau, sur la dégénérescence algébrique des courbes entières dans une hypersurface générique de $\mathbb{P}^N$ combiné avec un petit lemme de déplacement. Plus précisément, dans [DMR10], Diverio, Merker et Rousseau démontrent que pour une hypersurface générique de grand degré $H \subset \mathbb{P}^N$ il existe un sous-ensemble algébrique $Y \subset H$ contenant toutes les courbes entières de H. Un peu plus tard, Diverio et Trapani [DT09] démontrent que le lieu Y est de codimension au moins deux dans H. Considérons l'intersection de deux telles hypersurfaces H_1 et H_2 contenant des lieux $Y_1 \subset H_1$ et $Y_2 \subset H_2$ comme précédemment. Les courbes entières de $X := H_1 \cap H_2$ sont alors, bien entendu, contenues dans $Y_1 \cap Y_2$. Il reste alors à voir que quitte a déplacer légèrement H_1 et H_2 on peut s'assurer que $Y_1 \cap Y_2$ est de codimension au moins quatre dans $H_1 \cap H_2$. Ainsi, par intersections successives, on peut réduire la dimension de ce mauvais lieu jusqu'à obtenir hyperbolicité. D'autre part, nous observerons aussi que ce résultat initialement démontré pour les hypersurfaces génériques de grand degré dans $\mathbb{P}^N$ s'étend facilement aux hypersurfaces génériques suffisamment amples dans une variété projective lisse quelconque. Notre résultat s'énonce comme suit.

Théorème E. *Soit M une variété projective lisse de dimension N. Soient $A_1, \ldots, A_c$ des diviseurs amples sur M. Pour $d_1, \ldots, d_c \in \mathbb{N}$ suffisamment grands, prenons des hypersurfaces génériques $H_1 \in |d_1A_1|, \ldots, H_c \in |d_cA_c|$ et posons $X = H_1 \cap \cdots \cap H_c$. Alors il existe un sous-ensemble algébrique $Z \subset X$ de codimension au moins $2c$ qui contient toutes les courbes entières de X. En particulier, quand $2c \geqslant \dim X$, X est hyperbolique.*

Un point important concerne la notion de généricité dans cet énoncé. En effet, pour obtenir un résulat vrai génériquement, il est important d'exploiter le fait que le mauvais lieu construit par Diverio, Merker et Rousseau varie de façon algébrique avec l'hypersurface que l'on considère.

Chapitre 7

Dans le chapitre 7 on considère deux problèmes différents. Le premier concerne l'amplitude de $\Omega_X(2)$ et le second concerne la construction explicite de formes différentielles symétriques sur des variétés intersection complète particulières.

Le premier point est motivé par le fait que si $X \subseteq \mathbb{P}^N$ alors $\Omega_X(2)$ est globalement engendré. Il est alors naturel de se demander pour quelles variétés $X \subseteq \mathbb{P}^N$ le fibré $\Omega_X(2)$ est ample. En fait, l'amplitude de $\Omega_X(2)$ est caractérisée par la non-existence de droites dans X.

Théorème F. *Soit $X \subseteq \mathbb{P}^N$. Alors $\Omega_X(2)$ est ample si et seulement si X ne contient pas de droite.*

La démonstration de se résultat est basée sur l'interprétation de $\mathbb{P}(\Omega_X(2))$ comme étant la variété d'incidence « point-droite ». Cette interprétation nous a été signalée par Frédéric Han.

Le deuxième point concerne la construction explicite de formes différentielles symétriques sur les variétés intersection complète. Pour cela, nous utilisons à des fins calculatoires le fibré $\widetilde{\Omega}_X$ qui, *grosso modo*, est le faisceau des formes différentielles sur le cône de X qui descendent sur X mais qui ne vérifient pas nécessairement la condition d'Euler. Ce fibré a été étudié en particulier par Bogomolov et De Oliveira [BDO08] mais aussi par Debarre [Deb05]. Ce fibré n'est jamais ample mais Debarre a démontré que sous les hypothèses de sa conjecture le fibré $\widetilde{\Omega}_X(1)$ est ample. Il s'avère que ce fibré est en un certain sens mieux adapté au calcul que le fibré cotangent. Ceci nous permet de calculer explicitement des formes différentielles symétriques sur des variétés intersection complète particulières, et aussi d'obtenir des théorèmes d'annulation générique. Nous étudions dans un premier temps la cohomologie des fibrés $S^k\widetilde{\Omega}_X \otimes \mathcal{O}_X(-a)$ ce qui nous permet de donner quelques théorèmes d'annulation puis de déduire une stratégie de calcul. Nous illustrons ensuite cette stratégie en construisant (entre autres) un exemple de famille de surfaces intersection complète qui a des propriétés intéressantes.

Exemple. *Soient $e \geqslant 5$, $e_1 := \lfloor \frac{e}{2} \rfloor$ et $e_2 := \lceil \frac{e}{2} \rceil$ de sorte que $e = e_1 + e_2$. Pour $\alpha := (\alpha_1, \alpha_2) \in \mathbb{C}^2$ et $\beta := (\beta_1, \beta_2) \in \mathbb{C}^2$, posons*

$$\begin{aligned} F_\alpha &:= X_0^e + X_1^e + X_2^e + X_3^e + X_4^e + \alpha_1 X_0^{e_1} X_1^{e_2} + \alpha_2 X_2^{e_1} X_3^{e_2}, \\ G_\beta &:= a_0 X_0^e + a_1 X_1^e + a_2 X_2^e + a_3 X_3^e + a_4 X_4^e + \beta_1 X_0^{e_1} X_1^{e_2} + \beta_2 X_2^{e_1} X_3^{e_2}. \end{aligned}$$

Notons $S_\beta^\alpha := (F_\alpha = 0) \cap (G_\beta = 0) \subset \mathbb{P}^4$. Pour $(a_i)_{0 \leqslant i \leqslant 4} \in \mathbb{C}^5$ générique, ceci nous donne une famille de surfaces sur $\mathbb{P}^4$. Nous avons alors

- *$H^0(S_0^0, S^2\Omega_{S_0^0}) \neq 0$.*
- *Si $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{P}^4$ est générique, alors $H^0(S_\beta^\alpha, S^2\Omega_{S_\beta^\alpha}) = 0$.*

Nous voudrions signaler que nous pouvons décrire explicitement tous les éléments de $H^0(S_0^0, S^2\Omega_{S_0^0})$ et donner les équations du lieu base. En particulier ceci donne une nouvelle illustration de la non-invariance par déformation des nombres $h^0(X, S^m\Omega_X)$ (un phénomène déjà étudié dans [Bog78] et [BDO08]). Le travail que nous exposons dans ce chapitre est encore inachevé mais nous espérons qu'il présente déjà un certain intérêt.

Chapitre 8

Dans le chapitre 8, nous généralisons un théorème dû à Schneider [Sch92] concernant les puissances symétriques du fibré cotangent d'une variété projective. Notre résultat s'énonce comme suit.

Théorème G. *Soit $X \subseteq \mathbb{P}^N$ une variété projective lisse de dimension n et de codimension $c = N - n$. Considérons un entier $k \geqslant 1$ et k entiers $\ell_1, \ldots, \ell_k \geqslant 0$, et soit $a \in \mathbb{Z}$.*

1. *Si $\ell_1 + \cdots + \ell_k > a + k$ et $j < n - k \cdot c$, alors*
$$H^j(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$
2. *Si $\ell_1 + \cdots + \ell_k > a$ et $0 < n - k \cdot c$, alors*
$$H^0(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Schneider a démontré le cas $k = 1$ de ce théorème. Notre démonstration du théorème G est basée sur les mêmes idées que celles de [Sch92], c'est-à-dire l'utilisation de la positivité du fibré normal combinée avec de profonds théorèmes d'annulation. Cependant, là où Schneider utilise le théorème d'annulation de Le Potier, nous utilisons un théorème d'annulation plus général dû à Ein et Lazarsfeld [EL93] et Manivel [Man97]. À partir de cela, nous pouvons établir un théorème d'annulation pour les fibrés de différentielles de jets.

Théorème H. *Soit $X \subseteq \mathbb{P}^N$ une variété projective lisse de dimension n et de codimension c. Soit $a \in \mathbb{Z}$. Si $1 \leqslant k < \frac{n}{c}$ et $a < \frac{m}{k}$, alors*
$$H^0(X, E_{k,m}^{GG}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Ce résultat montre donc que l'existence d'équations différentielles de jets est une obstruction à l'existence de certains plongements dans l'espace projectif. Ceci est un analogue en ordre supérieur des observations de Schneider [Sch92]. Diverio [Div08] a déjà démontré un résultat dans cette direction. Le résultat de Diverio est exactement le théorème H dans le cas où X est une intersection complète et $a = 0$.
Pacienza et Rousseau ont utilisé les mêmes idées que Diverio pour démontrer un théorème d'annulation pour leurs fibrés de différentielles de jets généralisés sur des variétés intersection complète. Il n'est donc pas surprenant qu'en utilisant le théorème G on puisse se passer de l'hypothèse intersection complète. De plus nous obtenons aussi des résultats analogues pour les groupes de cohomologie d'ordres supérieurs. Ceci est le contenu du corollaire 8.4.2.

Chapitre 9

Dans le chapitre 9 nous abordons une problématique un peu différente qui est naturellement apparue lors de l'étude de la cohomologie des puissances symétriques du fibré cotangent d'une variété intersection complète. La question est de déterminer la cohomologie des fibrés en droites sur les hypersurfaces universelles. Cette question est le cas le plus simple de la question originelle et elle est déjà non triviale. Nous n'avons d'ailleurs pas réussi à la résoudre à part dans des cas particuliers. Notre résultat peut s'énoncer ainsi.

Théorème I. *Soit $\mathcal{H} := \{(t,x) \in \mathbb{P}^{N_d} \times \mathbb{P}^1 \ / \ t(x) = 0\}$, où $\mathbb{P}^{N_d} := \mathbb{P}\left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^*\right)$. Alors, pour tous $m_1, m_2 \in \mathbb{Z}$ tous les morphismes apparaissant dans la suite exacte longue associée à la suite exacte courte de restriction*

$$0 \to \mathcal{O}_{\mathbb{P}^{N_d}\times\mathbb{P}^1}(m_1 - 1, m_2 - d) \to \mathcal{O}_{\mathbb{P}^{N_d}\times\mathbb{P}^1}(m_1, m_2) \to \mathcal{O}_{\mathcal{H}}(m_1, m_2) \to 0$$

sont de rang maximal.

Ceci nous permet de d'obtenir beaucoup d'informations sur les groupes de cohomologie de $\mathcal{O}_{\mathcal{H}}(m_1, m_2)$. La preuve repose sur un algorithme de Gauss un peu compliqué et des difficultés combinatoires font que nous n'avons pas réussi à généraliser ces résultats en dimension plus grande, même si cela paraît accessible.

Notations et conventions

Nous résumons les principales notations que nous utilisons dans cette thèse. Elle sont toutes relativement standard.

- Nous travaillons sur le corps des nombres complexes $\mathbb{C}$.
- $\mathbb{P}^N$ désigne l'espace projectif de dimension N.
- Presque toujours, X désignera une variété projective lisse de dimension n.
- TX est le fibré tangent de X.

- $\Omega_X = TX^*$ est le fibré cotangent de X.
- $K_X = \det(\Omega_X)$ le diviseur canonique.
- $E_{k,m}^{GG}\Omega_X$ est le fibré des différentielles de jets de Green-Griffiths d'ordre k et de degré m (voir section 1.4.2).
- $E_{k,m}\Omega_X$ est le fibré des différentielles de jets de Demailly-Semple d'ordre k et de degré m (voir section 1.4.3).
- Dans ce qui suit E est un fibré vectoriel sur X.
- $\mathbb{P}(E) \to X$ désigne le projectivisé des *quotients de rang 1* de E et $\mathcal{O}_{\mathbb{P}(E)}(1)$ est le fibré tautologique sur $\mathbb{P}(E)$.
- $P(E) \to X$ désigne le projectivisé des *droites* dans E et $\mathcal{O}_{P(E)}(-1)$ est le fibré tautologique sur $P(E)$.
- $c_i(E)$ désigne la i-ème classe de Chern de E.
- $s_i(E)$ désigne la i-ème classe de Segre de E.
- Si L est un fibré en droites sur X, on définie le lieu base stable de L par

  $$\mathrm{Bs}(L) := \bigcap_{\substack{m>0 \\ s \in H^0(X,mL)}} (s=0),$$

  où $(s=0)$ est le lieu des zéros de la section s.
- Soit L est un fibré en droites ample sur X. Nous dirons qu'une propriété P est vraie pour un élément *générique* de $|L|$, s'il existe un ouvert de Zariski non vide $U \subseteq |L|$ tel que la propriété P soit vérifiée par tout élément de U.
- Pour $N > 0$ et $d \geqslant 0$ nous définissons $\mathbb{P}^{N_d} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}))$. À tout élément $t = (t_1, \ldots, t_c) \in \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_c}$ correspond la variété $X_t \subset \mathbb{P}^N$ qui a pour équations $t_1 = 0, \ldots, t_c = 0$. Nous dirons qu'une propriété P est vraie pour une intersection complète *générique* de $\mathbb{P}^N$, de codimension c et de multidegré $(d_1, \ldots, d_c)$, s'il existe un ouvert de Zariski $U \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_c}$ tel que la propriété P soit vérifiée par X_t quelque soit $t \in U$.

Chapter 1

Preliminaries

In this chapter we sum up briefly the main notions we use. Also, we give some motivations for the problems we study. We start by recalling the main notions of positivity for line bundles and some of their properties. After that we recall how to generalize these notions to vector bundles. Then we recall some basic facts concerning hyperbolicity, jet differentials and the positivity of the cotangent bundle. We conclude this chapter by some arithmetical motivations over number fields and function fields.

1.1 Positivity notions for line bundles

This section is mainly based on the books of R. Lazarsfeld [Laz04a] and [Laz04b]. We recall the main positivity notions we use (ampleness, nefness, bigness...).
For all this section we denote by X a connected projective variety, and by L a line bundle on X. We assume that X is smooth so that we can identify in the usual way line bundles, Cartier divisors and Weil divisors (much of what follows also makes sense when X is not smooth, see [Laz04a]).

1.1.1 Definitions

We suppose that $H^0(X, L) \neq 0$. One can define a rational map

$$\begin{array}{rcl} \varphi_L : X & \dashrightarrow & \mathbb{P}(H^0(X, L)) \\ x & \mapsto & \{s \in H^0(X, L) \ / \ s(x) = 0\}. \end{array}$$

Another way (maybe more intuitive) of defining the same map is the following : let $s_0, \dots, s_N$ be a basis of $H^0(X, L)$. Using this basis we identify $\mathbb{P}(H^0(X, L))$ with $\mathbb{P}^N$. Then we can write

$$\begin{array}{rcl} \varphi_L : X & \dashrightarrow & \mathbb{P}^N \\ x & \mapsto & [s_0(x) : \cdots : s_N(x)]. \end{array}$$

There is an abuse of notation in this last expression since $s_i(x)$ is not a number but an element in a line bundle so to give it a meaning one has to choose a trivialization for L in a neighborhood of x. But one can see that the element $[s_0(x) : \cdots : s_N(x)]$ does not depend on the choice of a trivialization. The main positivity notions that we need are the followings:

Definition 1.1.1. With the same notation, the line bundle L is:

- *very ample* if φ_L is an embedding,
- *very big* if φ_L is birational onto its image,
- *ample* if mL is very ample for some positive integer m,

- *big* if mL is very big for some positive integer m.

We then extend these notions to divisors using the standard identifications. In some sense, those notions are "open". The idea is now to construct the corresponding "closed" notions. Let us make this precise. We denote by $\mathrm{Div}_{\mathbb{R}}(X) = \mathrm{Div}(X) \otimes \mathbb{R}$ the space of $\mathbb{R}$-divisors. We recall that the *Néron-Severi group* is defined to be $N^1(X) := \mathrm{Div}_{\mathbb{R}}(X)/\equiv_{num}$ where $D_1 \equiv_{num} D_2$ if and only if $D_1 \cdot C = D_2 \cdot C$ for all curves C. It is a theorem that $N^1(X)$ is a finite dimensional vector space over $\mathbb{R}$. If D is a divisor on X, we denote by $[D]$ its class in $N^1(X)$; also we let $[L]$ be the class of L in $N^1(X)$ (this is well defined).

We then define the following cones in $N^1(X)$.

- $\mathrm{Amp}(X) \subseteq N^1(X)$ the cone spanned by the classes of ample divisors.
- $\mathrm{Big}(X) \subseteq N^1(X)$ the cone spanned by the classes of big divisors.

One of the notable properties of these cones is the following.

Proposition 1.1.2. $\mathrm{Amp}(X)$ *and* $\mathrm{Big}(X)$ *are open cones.*

We can now define the corresponding closed positivity notions. Let

- $\mathrm{Nef}(X) := \overline{\mathrm{Amp}(X)}$,
- $\mathrm{Pseff}(X) := \overline{\mathrm{Big}(X)}$.

Definition 1.1.3. With the above notation, L is:

- *nef* if $[L] \in \mathrm{Nef}(X)$,
- *pseudoeffective* if $[L] \in \mathrm{Pseff}(X)$.

1.1.2 Main properties

We now give the main results concerning those positivity notions. We will stay elementary and leave aside many important results. We refer to [Laz04a] for more details, and proofs. A remarkable fact is that ampleness can be characterized in several different manners, cohomologically, numerically and metrically. We start with the cohomological characterization.

Theorem 1.1.4 (Cartan-Serre-Grothendieck theorem)**.** *With the above notation, the following properties are equivalent:*

1. *L is ample.*
2. *Given any coherent sheaf $\mathcal{F}$ on X, there exists $m \in \mathbb{N}$ such that $\forall i > 0$, $\forall k \geqslant m$,*
$$H^i(X, \mathcal{F} \otimes L^k) = 0.$$
3. *Given any coherent sheaf $\mathcal{F}$ on X there exists $m \in \mathbb{N}$ such that $\mathcal{F} \otimes L^k$ is generated by its global sections for all $k \geqslant m$.*
4. *There exists $m \in \mathbb{N}$ such that L^k is very ample for all $k \geqslant m$.*

Now we recall the numerical characterization of ampleness.

Theorem 1.1.5 (Nakai-Moishezon-Kleiman criterion)**.** *With the above notation, L is ample if and only if*

$$\int_V c_1(L)^{\dim(V)} > 0$$

for every positive-dimensional irreducible subvariety $V \subseteq X$.

It should be noted that to check ampleness numerically, we have to check the above inequality for every subvariety of X. However, it is possible to check nefness by only looking at curves.

Theorem 1.1.6 (Kleiman theorem)**.** *With the above notation, L is nef if and only if*

$$\int_C c_1(L) \geqslant 0$$

for every irreducible curve C in X.

This is really useful as in general it is much easier to understand just the curves in X rather than all the higher dimensional subvarieties of X. We mention that these positivity notions have important analytic properties. A nice reference for this is [Dem01].

1.2 Positivity notions for vector bundles

There is some sort of canonical way to construct a positivity notion for vector bundles from a positivity notion for line bundles. However we will see that the situation is more complicated in this framework and we will lose some of the theorems that hold for line bundles.
In this section we take a projective variety X and a vector bundle E on X. We consider the projectivization $\pi_E : \mathbb{P}(E) \to X$ and the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$.

1.2.1 Ampleness and nefness

We use the above notation. We start by giving the main definition.

Definition 1.2.1. We say that:

- E is *ample* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle on $\mathbb{P}(E)$,
- E is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$.

With this definition we keep the nice cohomological properties of line bundles, but we lose the numerical and the metric characterization of ampleness. Since $\pi_{E*}\mathcal{O}_{\mathbb{P}(E)}(m) = S^m E$, the symmetric powers of the vector bundle replace the tensor powers of a line bundle in the properties. We begin with the cohomological characterization of ampleness due to Hartshorne [Har66], analogous to the Cartan-Serre-Grothendieck theorem.

Theorem 1.2.2. *With the above notation, the following are equivalent:*

1. *E is ample.*
2. *For any coherent sheaf $\mathcal{F}$ on X, there is a positive integer m such that for any $i > 0$ and any $k \geqslant m$*

$$H^i(X, S^k E \otimes \mathcal{F}) = 0.$$

3. *Given any coherent sheaf $\mathcal{F}$ on X there exists $m \in \mathbb{N}$ such that $\mathcal{F} \otimes S^k E$ is generated by its global sections for all $k \geqslant m$.*
4. *For any ample line bundle L, there exists $m \in \mathbb{N}$ such that for all $k \geqslant m$, $S^k E$ is the quotient of a direct sum of copies of L .*

Remark 1.2.3. Observe that in particular the nonvanishing of $H^0(X, S^k E)$ for some $k > 0$ can be seen as a first step towards ampleness.

Ample vector bundles satisfy some notable properties (as always we refer to [Laz04b] for the proofs).

Proposition 1.2.4. *Let Q be a vector bundle on X. If E is ample and if there is a surjective morphism of vector bundles*

$$E \to Q \to 0,$$

then Q is ample as well.

Proposition 1.2.5. *Let $f : Y \to X$ be a finite morphism. Then if E is ample, so is f^*E.*

Proposition 1.2.6. *Let F, G, and E be vector bundles on X such that there is an exact sequence of vector bundles,*

$$0 \to F \to E \to G \to 0.$$

If F and G are ample, then E is ample as well.

Those last three properties are also satisfied by nef vector bundles.

1.2.2 Numerical positivity

For vector bundles we no longer have a nice numerical criterion for ampleness as was shown by Fulton [Ful76]. But we can look for a numerical positivity notion for vector bundles.

Following Fulton [Ful98] we recall definitions concerning Schur polynomials. See also [Ful97] and [Man98]. Let $c_1, c_2, c_3, ...$ be a sequence of formal variables. Let ℓ be a positive integer and let $\lambda = (\lambda_1, ..., \lambda_\ell)$ be a partition of ℓ. We define the *Schur polynomial* associated with $c = (c_i)_{i\in\mathbb{N}}$ and λ as:

$$\Delta_\lambda(c) := \det\left[(c_{\lambda_i+j-i})_{1\leqslant i,j\leqslant \ell}\right].$$

For this to make sense we set $c_0 = 1$ and $c_i = 0$ for $i < 0$; this always holds in our applications. For example, $\Delta_1(c) = c_1$, $\Delta_{(2,0)}(c) = c_2$ and $\Delta_{(1,1)}(c) = c_1^2 - c_2$.

Now consider two sequences of formal variables, $c_1, c_2, c_3, \ldots$ and $s_1, s_2, s_3, \ldots$ satisfying the relation:

$$(1 + c_1t + c_2t^2 + \cdots) \cdot (1 - s_1t + s_2t^2 - \cdots) = 1. \tag{1.1}$$

Note that relation (1.1) is satisfied when $c_i = c_i(E)$ are the Chern classes of a vector bundle E over a variety X and $s_i = s_i(E)$ are its Segre classes.

The proof of the following crucial combinatorial result can be found in [Ful98].

Lemma 1.2.7. *With the same notation. Let $\bar{\lambda}$ be the conjugate partition of λ; then $\Delta_\lambda(c) = \Delta_{\bar{\lambda}}(s)$.*

Let E be a vector bundle of rank r over a projective variety X of dimension n.

Definition 1.2.8. We will say that E is numerically positive if for any subvariety $Y \subseteq X$ and for any partition λ of $\ell = \dim(Y)$ one has $\int_Y \Delta_\lambda(c(E)) > 0$.

This definition is motivated by a theorem of Fulton and Lazarsfeld [FL83] which gives numerical consequences of ampleness.

Theorem 1.2.9 (Fulton-Lazarsfeld)**.** *If E is ample then E is numerically positive. Moreover the Schur polynomials are exactly the relevant polynomials to test ampleness numerically.*

Note that the converse is false, for example the bundle $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$ is numerically positive but not ample (the problem being the lack of subvarieties of dimension two to test c_2). See [Ful76] for a more interesting example. This example also shows that the quotient of a numerical positive vector bundle is not necessarily numerically positive. However, one can remark the following, whose proof is immediate.

Proposition 1.2.10. *If E is a numerically positive vector bundle on a variety X and if Y is a subvariety of X then $E_{|Y}$ is numerically positive on Y.*

As we mentioned, numerical positivity is strictly weaker than ampleness. However, it seems interesting to know some special cases in which it is equivalent. For example in the case of line bundles, this is true by the Nakai-Moishezon-Kleiman criterion. One can give some other examples for higher-rank bundle.

Proposition 1.2.11. *Let X be a projective variety of dimension greater than two with Picard number* 1*. Let E be a rank-two vector bundle on X extension of two line bundles L_1 and L_2,*

$$0 \to L_1 \to E \to L_2 \to 0.$$

Then E is numerically positive if and only if E is ample.

Proof. Let α be an ample class in $N^1(X)$. Then $N^1(X) = \mathbb{R} \cdot \alpha$ and $\exists r_1, r_2 \in \mathbb{R}$ such that $[L_1] = r_1\alpha$ and $[L_2] = r_2\alpha$. Let C be a curve in X and S a surface in X. From Whitney's formula, we obtain

$$\int_C c_1(E) = \int_C (c_1(L_1) + c_1(L_2)) = (r_1 + r_2) \int_C \alpha$$

and

$$\int_S c_2(E) = \int_S c_1(L_1)c_1(L_2) = r_1 r_2 \int_S \alpha^2.$$

Now suppose that E is numerically positive. This implies that $\int_C c_1(E) > 0$ and $\int_S c_2(E) > 0$. Therefore $r_1 + r_2 > 0$ and $r_1 r_2 > 0$, and this implies $r_1 > 0$ and $r_2 > 0$. Thus L_1 and L_2 are ample, and this yields that E is ample as the extension of two ample line bundles. $\square$

1.2.3 Bigness

There are two natural ways of defining bigness for vector bundles depending on what control we want on the base locus of the tautological line bundle.

Definition 1.2.12. A vector bundle E is:

- *weakly big* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a big line bundle on $\mathbb{P}(E)$,
- *strongly big* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a big line bundle on $\mathbb{P}(E)$ and $\pi_E(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1)))$ is a proper algebraic subset of X.

In particular, E is weakly big if and only if there are positive constants $\alpha > 0$ and $\beta > 0$ such that for all $m \gg 0$,

$$\alpha \cdot m^{n+r-1} \leqslant h^0(X, S^m E) \leqslant \beta \cdot m^{n+r-1}.$$

We mention that what we call strong bigness is in general much more difficult to check than weak bigness. Those definitions are non-standard: often the term "big" is used for what we called "weakly big".

1.2.4 Almost everywhere ampleness

Almost everywhere ampleness is a notion due to Miyaoka [Miy83]. Here we follow very closely his presentation. Let E be a vector bundle on X. The main idea is to try to locate the "bad" curves in $\mathbb{P}(E)$. Fix an ample divisor H on X. Take $\epsilon \in \mathbb{R}$ and set

$$N_\epsilon := \{C \text{ irreducible curve in } \mathbb{P}(E) \mid \int_C c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) < \epsilon C \cdot H\},$$

and

$$M_\epsilon := \bigcup_{C \in N_\epsilon} \pi_E(C) \subseteq X.$$

Definition 1.2.13. Let $T \subset X$. A vector bundle E on X is said to be *ample modulo* T if $M_\epsilon \subseteq T$ for all sufficiently small positive numbers ϵ. If T can be chosen to be a proper algebraic subset then we say that E is *almost everywhere ample*.

Remark 1.2.14. This positivity notion is related to the ampleness and nefness thanks to the following remarks.

1. E is ample if and only if $N_\epsilon = \varnothing$ for some $\epsilon > 0$.

2. E is nef if and only if $N_0 = \varnothing$.

In Chapter 5 we will need the following, elementary, proposition.

Proposition 1.2.15. *Let E be a rank-r vector bundle on X. If E is ample modulo an algebraic subset of dimension* 0*, then E is ample.*

Proof. We have to prove that for all $\epsilon > 0$ small enough, $N_\epsilon = \varnothing$. Arguing by contradiction, suppose that there is an irreducible curve $C \subseteq \mathbb{P}(E)$ such that $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) < \epsilon C \cdot H$. By hypothesis, we know that $\pi_E(C) = \{x\}$ for some point x in X. Equivalently, this means that $C \subseteq \pi_E^{-1}(x)$. On the other hand, $\pi_E^{-1}(x) \simeq \mathbb{P}^{r-1}$, and under this isomorphism, $\mathcal{O}_{\mathbb{P}(E)}(1)_{|\pi_E^{-1}(x)} \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Therefore, since $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ is ample, we have $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) = \int_C c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1)) > 0$. But on the other hand, $C \cdot H = 0$. This yields a contradiction. $\square$

As shown by Miyaoka, almost everywhere ampleness is a “well-behaved” positivity notion.

Proposition 1.2.16. *Let E be an almost everywhere ample vector bundle on a projective variety X.*

1. *Any quotient of E is almost everywhere ample.*

2. *Let $f : Y \to X$ be a generically finite surjective morphism. Then f^*E is almost everywhere ample.*

Remark 1.2.17. We do not know if an extension of two almost everywhere ample vector bundles is also almost everywhere ample.

We would like to mention a property that makes a link between almost everywhere ampleness and bigness.

Proposition 1.2.18. *Let A be an ample line bundle on X. If $E \otimes A^{-1}$ is strongly big then E is almost everywhere ample.*

Proof. If a curve $C \subseteq \mathbb{P}(E)$ satisfies $\int_C c_1\left(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi_E^* A^{-1}\right) < 0$, then this curve has to lie in the base locus of $\mathcal{O}_{\mathbb{P}(E)(1)} \otimes \pi_E^* A^{-1}$ and therefore we conclude by the definition of strong bigness. $\square$

We would like to mention that some interesting work on ample and almost everywhere ample vector bundles on surfaces was done by Schneider-Tancredi ([ST85] and [ST88]).

1.2.5 Summary

We summarize the relations between all the notions defined above in a diagram.

$$
\begin{tikzcd}
& \text{Ample} \arrow[r] \arrow[dl] \arrow[d] \arrow[dr] & \text{Nef} \\
\text{Numerically Positive} & \text{Strongly Big} \arrow[d] & \text{A-e Ample} \\
& \text{Weakly Big} &
\end{tikzcd}
$$

1.3 Hyperbolicity

Hyperbolicity related problems are central in our work. Therefore, in this section, we give a brief account of this notion. We will also give some details on varieties whose cotangent bundle is ample. Several books and articles cover very nicely this topic, for example [Lan87], [Kob98]; there is also a chapter in [Laz04b]. Another important text is [Dem97a]. We also mention a survey article of Siu covering some modern techniques [Siu99].

1.3.1 Definitions and examples

Definition 1.3.1. A projective variety X is *hyperbolic* if every entire curve $f : \mathbb{C} \to X$ is constant.

Remark 1.3.2. We take here as a definition the so-called Brody hyperbolicity. This coincides with the usual Kobayashi hyperbolicity in our situation (X is compact) thanks to Brody's Theorem.

The first well-known hyperbolicity type result is Liouville's Theorem for entire functions.

Theorem 1.3.3. *(Liouville) Let $\Delta := \{z \in \mathbb{C} \ / \ |z| < 1\}$. If $f : \mathbb{C} \to \Delta$ is an entire function then f is constant.*

From this theorem one can directly deduce what are the hyperbolic algebraic curves. We use the classification theorem for curves. Let C be a smooth algebraic curve.

1. If $g(C) = 0$ then $C = \mathbb{P}^1$ which is not hyperbolic as it contains many entire curves.

2. If $g(C) = 1$ then $C = \mathbb{C}/\Lambda$ is an elliptic curve that comes with a non constant map $\mathbb{C} \to C$, therefore C is not hyperbolic.

3. If $g(C) \geqslant 2$ then C is hyperbolic. The universal cover of C is Δ. Therefore an entire curve $f : \mathbb{C} \to C$ yields a entire function $\tilde{f} : \mathbb{C} \to \Delta$ which has to be constant by Liouville's Theorem.

We see that the hyperbolicity of an algebraic curve is ruled by its genus in a very simple way. However, it is in general a very difficult question to know whether an algebraic variety of higher dimension is hyperbolic or not. There is an important conjecture which would give many examples.

Conjecture (Kobayashi)**.** *A generic hypersurface in $\mathbb{P}^N$ of degree big enough is hyperbolic.*

It is now known that Kobayashi's conjecture holds up to $N = 4$ by [DMR10] and [DT09]. In the last decades, many explicit families of hyperbolic hypersurfaces in $\mathbb{P}^N$ were constructed. When we refer to a Fermat type hypersurface, we mean a hypersurface whose equation is

$$X_0^d + \cdots + X_N^d = 0.$$

The explicit examples are mainly obtained by deforming slightly a Fermat type hypersurface. Such constructions were achieved by various means in the following works: Brody-Green [BG77], Masuda-Noguchi [MN96], Nadel [Nad89], Shirosaki [Shi98], Siu-Yeung [SY97], Demailly-El Goul [DEG00].
Another interesting work concerning the Fermat hypersurfaces was done by Green [Gre75]. We will say more about this conjecture in the section concerning jet differentials.

1.3.2 Ampleness of the cotangent bundle

Our main interest is the positivity of the cotangent bundle of a given variety. Varieties with ample cotangent bundle have many very interesting properties. However relatively few concrete examples are known. The knowledge of such examples could have some arithmetical applications as it would give concrete varieties on which we could test the arithmetical conjectures. This section is strongly inspired by [Laz04b].

Some consequences

Suppose that X is a projective variety whose cotangent bundle is ample. Then there are some easy remarks that one can make. First X is of general type, since $K_X = \det(\Omega_X)$ is ample as well. Moreover if $Y \subseteq X$ is a smooth subvariety then Ω_Y is also ample. This comes from the exact sequence

$$0 \to N^*_{X/Y} \to \Omega_{X_{|Y}} \to \Omega_Y \to 0$$

and the fact that ampleness is stable under restriction and quotient. In particular X does not contain rational curves, elliptic curves or abelian varieties. But more importantly to us, we have the following property, which is now a well-known fact.

Proposition 1.3.4 (Kobayashi)**.** *If Ω_X is ample, then X is hyperbolic.*

Schneider [Sch92] noticed that the existence of symmetric differential forms imposes strong restrictions on projective embeddings. As a particular case of his work, he proved the following.

Theorem 1.3.5 (Schneider)**.** *If $X \subseteq \mathbb{P}^N$ is an n-dimensional variety with ample cotangent bundle then $2n \leqslant N$.*

Known examples

As we said before, there are not many examples of varieties with ample cotangent bundle even though they are supposed to be relatively abundant. Let us recall some of the known examples.

Kodaira surfaces. In [Sch86], Schneider proved among other things that Kodaira surfaces have ample cotangent bundle.

Surfaces with positive indices. Miyaoka [Miy83], based on previous results [Miy77], proved that if a surface satisfies $c_1^2 > 2c_2$ then its cotangent bundle is almost everywhere ample. Building on ideas of Bogomolov, he then considered two such surfaces S_1 and S_2, and he constructed examples of surfaces with ample cotangent bundle as complete intersections of two generic divisors in $S_1 \times S_2$.

Branched cover over line arrangements. In [Hir83] Hirzebruch constructed and studied some desingularizations of surfaces covering of $\mathbb{P}^2$ branched along some arrangements of lines. Sommese [Som84] classified exactly the line arrangements that yield surfaces with ample cotangent bundle.

Complete intersection in a product of varieties with big cotangent bundle. Bogomolov constructed varieties with ample cotangent bundle as complete intersections in a product of varieties having big cotangent bundle. This construction can be found in [Deb05].

Complete intersection in abelian varieties. This is the example we are most interested in. In [Deb05] Debarre proved the following theorem.

Theorem 1.3.6 (Debarre)**.** *Let A be an N-dimensional abelian variety. Then the intersection of at least $\frac{N}{2}$ sufficiently ample generic divisors in A has ample cotangent bundle.*

Motivated by this result he conjectures that the analogous result should hold in projective spaces as well.

Conjecture 1.3.7 (Debarre)**.** *The intersection of at least $\frac{N}{2}$ sufficiently ample generic hypersurfaces in $\mathbb{P}^N$ has ample cotangent bundle.*

This conjecture is the starting point of our work. Therefore let us point out some of the difficulties that appear.
Note that in all the complete intersection type examples, the ambient variety comes with some positivity for the cotangent bundle. As we said before, the positivity of the cotangent bundle can only increase under

restriction, therefore one has some sort of positivity to start with. Even in the case of abelian varieties, the cotangent bundle Ω_A is trivial, and therefore nef.
However in the case of projective space there is no such thing: $\Omega_{\mathbb{P}^N}$ is negative, in the sense that $T\mathbb{P}^N$ is ample. Moreover, it should be noted that such a complete intersection has no global holomorphic differential forms (we refer to chapter 7), and therefore the cotangent bundle will never be globally generated.

1.4 Jet bundles and jet differentials

Since the work of Green and Griffiths [GG80], the theory of jet differentials has developed into a very powerful tool to study hyperbolicity. The different jet differential bundles play a central role in our work, thus we give here a brief account, very much inspired by Demailly's notes [Dem97a]. We will however adopt here the "dual" approach of [Mou09] since it is better suited to our computations. Another reference is Merker's paper [Mer10] where many details are carried out explicitly.

1.4.1 Jets of curves

Let X be a projective variety of dimension n. For all $k \geqslant 1$ we denote by $J_kX \stackrel{p_k}{\to} X$ the holomorphic bundle of k-jets of germs of holomorphic curves $f : (\mathbb{C}, 0) \to X$, that is $J_kX := \{f : (\mathbb{C}, 0) \to X\}/\sim$, where two germs $f, g : (\mathbb{C}, 0) \to X$ are equivalent ($f \sim g$) if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $0 \leqslant j \leqslant k$. The projection is then simply defined by

$$\begin{array}{ccc} J_kX & \stackrel{p_k}{\to} & X \\ f & \mapsto & f(0). \end{array}$$

Those spaces naturally have the structure of holomorphic fiber bundles over X, but unless $k = 1$ they are not vector bundles. Moreover, for any $x \in X$, $J_kX_x \cong (\mathbb{C}^n)^k$. To see this take local coordinates $(z_1, \ldots, z_n)$ around $x \in X$; we can then write $f = (f_1, \ldots, f_n)$ and the k-jet will then be entirely determined, by Taylor's formula, by

$$(f_1'(0), \ldots, f_n'(0), f_1''(0), \ldots, f_n''(0), \ldots, f_1^{(k)}(0), \ldots, f_n^{(k)}(0)).$$

1.4.2 Green-Griffiths jet differential bundles

With the above notation, for each $k \geqslant 1$, there is a natural $\mathbb{C}^*$-action on J_kX. Namely if $\lambda \in \mathbb{C}^*$ and $f : (\mathbb{C}, 0) \to (X, x)$ then

$$\begin{array}{rcl} \lambda \cdot f : (\mathbb{C}, 0) & \to & (X, x) \\ t & \mapsto & f(\lambda t). \end{array}$$

This action is easily expressed fiberwise. Namely, if one considers local coordinates around $x = f(0) = \lambda \cdot f(0)$ then f is represented by $(f'(0), f''(0), \ldots, f^{(k)}(0))$ and

$$\lambda \cdot \left(f'(0), f''(0), \ldots, f^{(k)}(0)\right) = \left(\lambda f'(0), \lambda^2 f''(0), \ldots, \lambda^k f^{(k)}(0)\right).$$

Then the Green-Griffiths jet differentials are defined as follows. Fiberwise we consider

$$E_{k,m}^{GG}\Omega_{X,x} := \left\{Q \in \mathbb{C}[f', \ldots, f^{(k)}] \ / \ \forall \lambda \in \mathbb{C}^*, \ Q\left(\lambda \cdot \left(f', \ldots, f^{(k)}\right)\right) = \lambda^m Q\left(f', \ldots, f^{(k)}\right)\right\}.$$

One can make this even more explicit: when $x \in X$ is fixed, one can consider coordinates

$$(f_1', \ldots, f_n', \ldots, f_1^{(k)}, \ldots, f_n^{(k)})$$

on J_kX_x. Then an element $Q \in E_{k,m}^{GG}\Omega_{X,x}$ is exactly a polynomial in the variables $(f_i^{(j)})$ of the form

$$Q = \sum_{\substack{I_1,\dots,I_k \in \mathbb{N}^n \\ |I_1|+2|I_2|+\cdots+k|I_k|=m}} a_{I_1,\dots,I_k}(f')^{I_1}\cdots(f^{(k)})^{I_k},$$

where we use the standard multi-index notation for $I = (i_1, \dots, i_n)$ and $1 \leqslant j \leqslant k$ we set $\left(f^{(j)}\right)^I := \left(f_1^{(j)}\right)^{i_1} \cdots \left(f_n^{(j)}\right)^{i_n}$. It turns out that these fibers can be arranged into a vector bundle over X.
The bundle $E_{k,m}^{GG}\Omega_X$ admits a natural filtration. We briefly recall its construction; however, we will not go into details, in particular we don't justify why everything is well-defined.
Fix local coordinates around $x \in X$ as above. For each $p \in \mathbb{N}$ and for each $1 \leqslant s \leqslant k$ define

$$F_s^p = F_s^p(E_{k,m}^{GG}\Omega_{X,x}) = \left\{ \begin{array}{c} Q \in E_{k,m}^{GG}\Omega_{X,x} \text{ involving only monomials } (f')^{I_1}\cdots(f^{(k)})^{I_k} \\ \text{with } |I_1|+2|I_2|+\cdots+s|I_s| \geqslant p \end{array} \right\}.$$

This gives a filtration

$$\{0\} = F_s^{m+1} \subseteq F_s^m \subseteq \cdots \subseteq F_s^1 \subseteq F_s^0 = E_{k,m}^{GG}\Omega_X.$$

We consider the associated graded terms

$$\mathrm{Gr}_s^p = \mathrm{Gr}_s^p(E_{k,m}^{GG}\Omega_X) := F_s^p/F_s^{p+1}.$$

Observe that

$$\begin{aligned} \mathrm{Gr}_{k-1}^p &\cong \left\{ \begin{array}{c} Q \text{ involving only monomials } (f')^{I_1}\cdots(f^{(k)})^{I_k} \\ \text{with } |I_1|+2|I_2|+\cdots+(k-1)|I_{k-1}| = p \end{array} \right\} \\ &\cong \left\{ \begin{array}{c} Q \text{ involving only monomials } (f')^{I_1}\cdots(f^{(k)})^{I_k} \\ \text{with } k|I_k| = m-p \end{array} \right\}. \end{aligned}$$

Therefore $\mathrm{Gr}_{k-1}^p \neq 0$ if and only if there is an integer $\ell_k \in \mathbb{N}$ such that $p = m - k\ell_k$. Whenever this is satisfied, if one looks closely at the coordinate changes, one observe that

$$\mathrm{Gr}_{k-1}^p = \mathrm{Gr}_{k-1}^{m-k\ell_k} \cong E_{k-1,m-k\ell_k}^{GG}\Omega_X \otimes S^{\ell_k}\Omega_X.$$

And therefore,

$$\mathrm{Gr}_{k-1}^\bullet = \bigoplus_{1\leqslant p\leqslant m} \mathrm{Gr}_{k-1}^p \cong \bigoplus_{0\leqslant \ell_k \leqslant \lfloor \frac{m}{k} \rfloor} E_{k-1,m-k\ell_k}^{GG}\Omega_X \otimes S^{\ell_k}\Omega_X.$$

Combining all those filtrations, we find inductively a filtration $F^\bullet$ on $E_{k,m}^{GG}\Omega_X$ such that the associated graded bundle is

$$\mathrm{Gr}^\bullet(E_{k,m}^{GG}\Omega_X) = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X.$$

1.4.3 Demailly-Semple jet differential bundles

There is another relevant bundle one can work with. This theory was developed by Demailly [Dem97a]. Consider the group of k-jets of biholomorphisms of $(\mathbb{C}, 0)$,

$$\mathbb{G}_k := \{\varphi : t \mapsto a_1 t + \frac{a_2}{2!}t^2 + \cdots + \frac{a_k}{k!}t^k + O(t^{k+1}) \text{ with } \mathrm{a}_1 \in \mathbb{C}^*\}.$$

Then one has a natural action from $\mathbb{G}_k$ on J_kX by reparametrization. Namely, if $\varphi \in \mathbb{G}_k$ and $f : (\mathbb{C}, 0) \to X$ we set $\varphi \cdot f := f \circ \varphi$. The Demailly-Semple jet differential bundle is defined as follows. Consider fiberwise the subspace

$$E_{k,m}\Omega_{X,x} := \left\{ Q \in E_{k,m}^{GG}\Omega_{X,x} \ / \ \forall \varphi \in \mathbb{G}_k,\ Q\left(\varphi \cdot \left(f', \cdots, f^{(k)}\right)\right) = (\varphi'(0))^m Q\left(f', \cdots, f^{(k)}\right) \right\}.$$

It turns out that those subspaces can be arranged into a subbundle of $E_{k,m}^{GG}\Omega_X$. There is a more geometric construction for obtaining the invariant jet differential bundles, which we shall now describe.

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. For all $k \in \mathbb{N}$ we can construct a variety X_k and a rank-n vector bundle $\mathcal{F}_k$ on X_k. Inductively, $X_0 := X$ and $\mathcal{F}_0 := \Omega_X$. Let $k \geqslant 0$ and suppose that X_k and $\mathcal{F}_k$ are constructed. Then $X_{k+1} := \mathbb{P}(\mathcal{F}_k) \overset{\pi_{k,k+1}}{\longrightarrow} X_k$ and $\mathcal{F}_{k+1}$ is the quotient of $\Omega_{X_{k+1}}$ defined by the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & S_{k+1} \arrow[r, equal] \arrow[d] & S_{k+1} \arrow[d] & & \\
0 \arrow[r] & \pi^*_{k,k+1}\Omega_{X_k} \arrow[r] \arrow[d] & \Omega_{X_{k+1}} \arrow[r] \arrow[d] & \Omega_{X_{k+1}/X_k} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_{k+1}}(1) \arrow[r] \arrow[d] & \mathcal{F}_{k+1} \arrow[r] \arrow[d] & \Omega_{X_{k+1}/X_k} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

For all $k > j \geqslant 0$ we will set $\pi_{j,k} = \pi_{j,j+1} \circ \cdots \circ \pi_{k-1,k} : X_k \to X_j$, $\pi_k := \pi_{0,k}$. We then have

Theorem 1.4.1 ([Dem97a])**.** *Let A be an ample linde bundle on X. Then*

1. *$E_{k,m}\Omega_X \simeq \pi_{k*}\mathcal{O}_{X_k}(m)$.*
2. *$H^0(X, E_{k,m}\Omega_X \otimes A^{-1}) \simeq H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^* A^{-1})$.*

Note that $n_k := \dim(X_k) = n + k(n-1)$. If we have a k-uple of integers $(a_1, \dots, a_k)$, we will write

$$\mathcal{O}_{X_k}(a_1, \dots, a_k) = \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k).$$

This tower construction is better understood with the following picture.

$$\begin{tikzcd}
\mathcal{F}_k \arrow[dr, bend left] & \vdots \arrow[d] & \mathcal{O}_{X_k}(1) \arrow[dl, bend right] \\
 & X_k \arrow[d, "\pi_{k-1,k}"] & \\
\mathcal{F}_2 \arrow[dr, bend left] & \vdots \arrow[d] & \mathcal{O}_{X_2}(1) \arrow[dl, bend right] \\
\mathcal{F}_1 \arrow[dr, bend left] & X_2 \arrow[d] & \mathcal{O}_{X_1}(1) \arrow[dl, bend right] \\
\Omega_X \arrow[dr, bend left] & X_1 \arrow[d, "\pi_{0,1}"] & \\
 & X &
\end{tikzcd}$$

1.4.4 The vanishing criterion

The theorem that makes the link between these bundles and hyperbolicity is the following:

Theorem 1.4.2. *Let A be an ample line bundle on X. Then for any entire curve $f : \mathbb{C} \to X$ and all $P \in H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$ one has $P(f_{[k]}) \equiv 0$. This also holds for any $P \in H^0(X, E^{GG}_{k,m}\Omega_X \otimes A^{-1})$.*

Many authors contributed to prove this theorem: Green-Griffiths [GG80], Demailly [Dem97a] for the invariant case, Siu-Yeung [SY97] for the general case. We should also mention the existence of another proof due to Yamanoi [Yam04], and also that a result for jets of order 1 was obtained previously by Noguchi [Nog77]. We also refer to [Dem97b] for a nice account of Siu-Yeung's appraoch.

This theorem justifies much of the work that we do in the sequel, as it points out that the existence of such jet differential equations have strong applications in controlling the images of entire curves. This theorem also contains in its core the whole analytic information. This is why we are able to study transcendental objects by purely algebraic means.

1.4.5 Pacienza-Rousseau generalized jet differential bundles

The Green-Griffiths jet differentials and the Demailly-Semple jet differentials were constructed to study entire maps $f : \mathbb{C} \to X$. Pacienza and Rousseau [PR08] generalized this construction to study more generally holomorphic maps $f : \mathbb{C}^p \to X$. For each $p \geqslant 1$, they constructed bundles $E^{GG}_{p,k,m}\Omega_X$ and $E_{p,k,m}\Omega_X$ generalizing $E^{GG}_{k,m}\Omega_X = E^{GG}_{1,k,m}\Omega_X$ and $E_{k,m}\Omega_X = E_{1,k,m}\Omega_X$. Their construction is very similar to the ones done above, however there are some unexpected difficulties appearing. We just really briefly recall the definitions, as we will just give one result on those bundles in Chapter 8. For the details we refer to [PR08].

Fix $p \geq 1$. The idea is to consider the space $J_{k,p}X$ of k-jets of germs of holomorphic maps $f : (\mathbb{C}^p, 0) \to X$. As above, this space comes with a natural $(\mathbb{C}^*)^p$-action. Namely, for $f : (\mathbb{C}^p, 0) \to X$ and $\lambda = (\lambda_1, \dots, \lambda_p) \in (\mathbb{C}^*)^p$ we define

$$\begin{array}{rcl} \lambda \cdot f : (\mathbb{C}^p, 0) & \to & X \\ (t_1, \dots, t_p) & \mapsto & f(\lambda_1 t_1, \dots, \lambda_p t_p). \end{array}$$

Then $E^{GG}_{p,k,m}\Omega_X$ and $E_{p,k,m}\Omega_X$ are defined as follows: for each $x \in X$ we consider

$$E^{GG}_{p,k,m}\Omega_{X,x} := \left\{ \begin{array}{c} Q(f', \dots, f^{(k)}) \ / \ Q(\lambda \cdot (f', \dots, f^{(k)}) = \lambda_1^m \cdots \lambda_p^m Q(f', \dots, f^{(k)}) \\ \text{for all } \lambda = (\lambda_1, \dots, \lambda_p) \in (\mathbb{C}^*)^p \end{array} \right\}$$

and

$$E_{p,k,m}\Omega_{X,x} := \left\{ \begin{array}{c} Q(f', \dots, f^{(k)}) \ / \ Q(\varphi \cdot (f', \dots, f^{(k)}) = (J_\varphi)^m Q(f', \dots, f^{(k)}) \\ \text{for all } \varphi \in \mathbb{G}_{p,k} \end{array} \right\},$$

where $\mathbb{G}_{p,k}$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$, and where J_φ denotes the Jacobian of φ. Those spaces can be arranged into vector bundles and $E_{p,k,m}\Omega_X$ is a subbunlde of $E^{GG}_{p,k,m}\Omega_X$. Pacienza and Rousseau proved, among other things, a vanishing criterion in this situation similar to the one when $p = 1$. However the only thing we will need concerning these bundles is that $E^{GG}_{p,k,m}\Omega_X$ admits a filtration whose graded terms are

$$\bigotimes_{\alpha \in I_1} S^{q^1_\alpha}\Omega_X \cdots \bigotimes_{\alpha \in I_k} S^{q^k_\alpha}\Omega_X$$

where

$$\sum_{\ell=1}^{k} \sum_{\alpha \in I_\ell} q^\ell_\alpha \alpha = (m, \dots, m),$$

and where for $\ell \in \mathbb{N}$, $I_\ell := \{\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p \ / \ \alpha_1 + \cdots + \alpha_p = \ell\}$. See Remark 2.6 in [PR08].

1.4.6 Algebraic degeneracy for generic hypersurfaces

A long-standing conjecture due to Kobayashi states that a generic hypersurface of sufficiently high degree in projective space should be hyperbolic. Using jet differentials, Y-T Siu ([Siu04]) gave a general strategy to attack this problem. This strategy was afterwards studied in more details by S. Diverio E. Rousseau and J. Merker in [DMR10]. They were not able to prove the full statement but they were able to prove (effective) algebraic degeneracy, and this was already a major breakthrough.
The naive idea is to apply the vanishing criterion on generic hypersurfaces in $\mathbb{P}^N$. To apply the vanishing criterion there are two important steps.

1. Construct (many) non zero elements of $H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$ (or $H^0(X, E^{GG}_{k,m}\Omega_X \otimes A^{-1})$) for some ample divisor A on X.

2. Study the base locus of all the constructed differential equations.

To solve step 1 it is tempting to use a Riemann-Roch computation to prove that $H^0(X, E_{k,m}\Omega_X \otimes A^{-1}) \neq 0$ but to do this one has to control the higher-order cohomology groups, which is very difficult. It was Diverio [Div08], [Div09] who first managed to avoid those computations. To do so, he used Demailly's holomorphic Morse inequalities. Step 2 is still in general a major issue: the problem is that even though we know that there are many differential equations, we don't have much information on them. It seems therefore difficult to study the base locus. Siu [Siu04] gave a strategy to control the base locus. His idea is to use some global generation property for the tangent bundle to the vertical jet space on the universal hypersurface. This global generation property was eventually proved in full generality by Merker [Mer09].

This strategy has various applications, therefore we would like to make a brief account of it, hoping to make the upcoming section easier to read. To see how this works in general consider the following situation. Let E be a vector bundle on a variety X and suppose there is a vector field $\xi \in H^0(X, TX)$ and a section $\sigma \in H^0(X, E)$. Differentiation yields a morphism $E \overset{d}{\to} E \otimes \Omega_X$. Therefore we get a section $d\sigma \in H^0(X, E \otimes \Omega_X)$ and we can now apply ξ to this section to get a new section $d_\xi \sigma \in H^0(X, E)$ (this section is also denoted $L_\xi \sigma$). Therefore we see how global vector fields may be used to create new sections.

However in general there is no such global vector field, therefore one can not apply this technique directly. As an illustration, consider a variety X with a vector bundle E and an ample line bundle L. Suppose there is a meromorphic vector field $\xi \in H^0(X, TX \otimes L)$. If we start with a section $\sigma \in H^0(X, E)$ then $d_\xi \sigma \in H^0(X, E \otimes L)$. Therefore if we want our "new" section to be a section of E we have to start with a section of $E \otimes L^{-1}$. This simple remark explains all the complicated twists that will appear during the proofs.

However, this will not be sufficient to prove what we want, basically because we will be in a situation in which one cannot have at the same time a section of $E \otimes L^{-1}$ and a meromorphic vector field in $TX \otimes L$. Instead we have to consider the situation in a family of varieties and the so-called "slanted" vector fields. Let us illustrate this. Suppose one has a family of varieties $\mathcal{X} \to B$ and a vector bundle E on $\mathcal{X}$, and that we are looking after sections of $E_t \to X_t$ for generic $t \in B$. Suppose also that we have an ample line bundle L on $\mathcal{X}$, a section $\sigma_t \in H^0(X_t, E_t \otimes L^{-1})$ that extends to a section $\sigma \in H^0(\mathcal{X}, E \otimes L^{-1})$, and that there is a vector field $\xi \in H^0(\mathcal{X}, T\mathcal{X} \otimes L)$. This yields $d_\xi \sigma \in H^0(\mathcal{X}, E)$ and by restriction this gives the desired section in $H^0(X_t, E_t)$. Here we want to use in a critical way the flexibility of being able to differentiate with respect to a vector field that is only tangent to $\mathcal{X}$ and not necessarily tangent to X_t.

The original application of this strategy is to approach Kobayashi's conjecture. In this situation the variety $\mathcal{X}$ is the variety of regular relative jets of order $N-1$ on the universal hypersurface of degree d, the vector bundle is the jet differential bundle of order $N-1$, and the base is an open subset of the parameter space of hypersurfaces of degree d in $\mathbb{P}^N$.

Let us start by introducing some notation. Let $N \in \mathbb{N}$. For $d \in \mathbb{N}$ we let $\mathbb{P}^{N_d} := P\left(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))\right)$. The universal hypersurface of degree d is denoted by

$$\mathcal{H}_d := \{(t,x) \in \mathbb{P}^{N_d} \times \mathbb{P}^N / t(x) = 0\}.$$

Let $\mathcal{F} := \Omega_{\mathcal{H}/\mathbb{P}^{N_d}}$. The vertical jet spaces of order k are then denoted by $J_k\mathcal{F}^*$ and the space of relative regular jets is denoted by $J_k^{reg}\mathcal{F}^*$, so that for any $t \in \mathbb{P}^{N_d}$ we have $J_k\mathcal{F}^*_{|H_t} = J_k H_t$ and $J_k^{reg}\mathcal{F}^*_{|H_t} = J_k^{reg} H_t$. We also need to consider the relative jet differential bundles $E_{k,m}\Omega_{\mathcal{H}/\mathbb{P}^{N_d}}$. This is where the notion of directed manifolds introduced by Demailly [Dem97a] is very useful.

With this notation, the proof goes (very roughly) as follows.

- Step 1. Prove that $TJ_{N-1}\mathcal{F}^* \otimes \mathcal{O}_{\mathbb{P}^N}(M_N) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(k)$ is globally generated on $J_{N-1}^{reg}\mathcal{F}^*$.
- Step 2. Prove a non vanishing theorem of the type $H^0(H, E_{N-1,m}\Omega_H \otimes A^{-1}) \neq 0$ for a smooth hypersurface of sufficiently high degree in $\mathbb{P}^N$.
- Step 3. Use upper semicontinuity to extend sections of $H^0(H_t, E_{N-1,m}\Omega_H \otimes A^{-1})$ to sections of $H^0(U, E_{N-1,m}\Omega_{\mathcal{H}/\mathbb{P}^{N_d}} \otimes A^{-1})$.
- Step 4. Differentiate this section (viewed as a polynomial on $J_{N-1}\mathcal{F}^*$) with respect to tangent vector fields constructed in step 1.
- Step 5. Restrict this new section to H_t.
- Step 6. Start again steps 3 to 5 until one can say something on the intersection of the zero locus of all the sections obtained.

Let us make some remarks. The key point in step 1 is that M_N does not depend on d. Because the chosen degree d depends on the value of A in step 2, and this A has to be chosen sufficiently big so that one can go through steps 3 to 5 enough times and still get information (because of the twist by $\mathcal{O}_{\mathbb{P}^N}(M_n)$ at each time we apply step 4!). We refer to [DMR10] for all the details.

To give the credits, Step 1 was successfully completed by Merker [Mer09], but we should mention other contributors. The first such result is due to Voisin [Voi96], afterwards generalized by Siu [Siu04]. Some particular cases were done by Rousseau [Rou06] and also Păun [Pău08]. Step 2 was successfully completed by Diverio [Div08], but again there were previous works by Rousseau and Păun.

1.5 Arithmetical motivations

One of the main motivations of those problems (apart from their intrinsic beauty) lies in possible applications to arithmetical problems. It is now well-understood that the geometry of a projective variety governs its arithmetics. There are many conjectures which predict the relations between rational points on a variety and entire curves.

1.5.1 The number field case

In this section we take a number field K and a projective variety X defined over K. The analogue of the Green-Griffiths conjecture in this setting is the following conjecture of Lang.

Conjecture (Lang). *If X is a variety of general type then $X(K)$ is not Zariski dense in X.*

Another conjecture that reflects the relations between entire curves and rational points is the following.

Conjecture (Lang). *$X(\mathbb{C})$ is hyperbolic if and only if for any finite extension L of K there are only finitely many L-rational points on X.*

In particular, a weaker form of this conjecture is:

Conjecture (Lang). *If $\Omega_{X(\mathbb{C})}$ is ample, then there are only finitely many K-rational points on X.*

Even if both conjectures are still widely open, we should mention two notable special cases. First the one-dimensional case, previously known as Mordell's conjecture, proven by Faltings [Fal83],

Theorem (Faltings). *If C is a curve of genus $g \geqslant 2$ defined over K, then $C(K)$ is finite.*

We should also mention that Moriwaki [Mor95b] (based on results of Faltings [Fal91]) proved that the second conjecture is true if the cotangent bundle is also globally generated.

Theorem (Moriwaki). *If $\Omega_{X(\mathbb{C})}$ is ample and globally generated, then $X(K)$ is finite.*

However it is a very strong hypothesis to ask that the cotangent bundle be globally generated. For example, the cotangent bundle of a complete intersection variety in $\mathbb{P}^N$ of dimension at least two is never globally generated.

We would like to mention also a work of Noguchi [Nog03]. Shirosaki [Shi98] considered the following situation. Let $e, d \in \mathbb{N}$ be two relatively prime integers such that $d > 2e + 8$, then define recursively,

$$\begin{aligned} P_1(X_0, X_1) &:= X_0^d + X_1^d + X_0^e X_1^{d-e} \\ P_N(X_0, \ldots, X_N) &:= P_{N-1}(P_1(X_0, X_1), \ldots, P_1(X_{N-1}, X_N)) \quad \text{for } N \geqslant 2. \end{aligned}$$

And let $X := (P_N = 0) \subset \mathbb{P}^N_{\mathbb{Q}}$. Shirosaki proved that $X(\mathbb{C})$ is hyperbolic when $e > 2$, and Noguchi proved that Lang's conjecture holds for this particular hypersurface.

Theorem (Noguchi). *In the above setting. For any number field K, $X(K)$ is finite.*

1.5.2 The function field case

It is also very interesting and very rich to study the arithmetics of function fields. The situation here is better understood as one can construct a more geometric framework. We follow here [Nog82] for the presentation of this framework. The setting is the following. Let k be an algebraically closed field of characteristic 0. Take a function field K over k and a smooth projective variety $\mathcal{X}$ defined over K. Take a smooth algebraic variety R over k whose function field is K. Then standard arguments show that one can construct a subvariety $X \subseteq R \times \mathbb{P}^N_k$ with a projection $\pi : X \to R$ whose generic fiber is $\mathcal{X}$. In this setting, K-rational points $\mathcal{X}(K)$ correspond to rational sections of $\pi : X \to R$. Therefore the study of rational points can be seen as the study of sections of some morphism, which gives more tools than in the number field situation. To simplify the exposition we will from now on take as usual $k = \mathbb{C}$.

In this setting Lang conjecture can therefore be translated as follows,

Conjecture (Lang). *Let $\pi : X \to Y$ be a projective surjective morphism of complex algebraic manifolds, whose generic fibers are of general type. If π is not birationally trivial, then there is a proper subscheme of X that contains the image of all sections of π.*

The one-dimensional case of this conjecture (Mordell's conjecture for function fields) was solved by Grauert [Gra65]. Under the ampleness of the cotangent bundle assumption, it was solved by Noguchi [Nog82]. For some related results, see also [MD84], [Nog85] and [Mor95a]. Because of the features it shares with our work, we would like to mention a recent result due to Mourougane [Mou09].

Theorem (Mourougane). *For a general moving enough family of high enough degree hypersurfaces in $\mathbb{P}^N$, there is a proper algebraic subset of the total space that contains the image of all sections.*

The proof of this result is a generalization of Grauert ideas to higher-order jets by jet differential techniques. We would like to stress that during the proof of this result, Mourougane made some intersection computations that already contained the main ideas of the computations we will make in Chapter 4.

Chapter 2

Intersection computations on complete intersection varieties

Intersection computations will play a crucial role in the upcoming chapters, therefore we spend a whole chapter detailing those computations. We start by computing the Segre classes of the cotangent bundle of a complete intersection variety. Then we compute the Segre classes of the different bundles $\mathcal{F}_k$ in the Demailly Semple jet tower construction. The systematic use of Segre classes instead of Chern classes highly simplifies the computations, as they naturally appear when one pushes down the Chern classes of the tautological line bundles.

2.1 Segre classes in the jet tower

Let us first recall the definition of the Segre classes associated to a vector bundle. If E is a rank-r complex vector bundle on X and $p : \mathbb{P}(E) \to X$ the projection, the Segre classes of E are defined by

$$s_i(E) := p_* c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+i}.$$

(Note that they are denoted by $s_i(E^*)$ in [Ful98]). It is straightforward to check that for any line bundle $L \to X$

$$s_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-1+i}{i-j} s_j(E) c_1(L)^{i-j}. \tag{2.1}$$

Recall that the total Segre class is the formal inverse of the total Chern class of the dual bundle: $s(E) = c(E^*)^{-1}$. Therefore total Segre classes satisfy Whitney's formula for exact sequences of vector bundles. We take the notation of section 1.4.3. We set $s_{k,i} := s_i(\mathcal{F}_k)$, $s_i := s_i(\Omega_X)$, $u_k := c_1(\mathcal{O}_{X_k}(1))$. Moreover if H is an ample line bundle on X we usually set $h := c_1(H)$, and $\mathcal{C}_k(X, H) := \mathbb{Z} \cdot u_k \oplus \cdots \oplus \mathbb{Z} \cdot u_1 \oplus \mathbb{Z} \cdot h \subset N^1(X_k)$. To ease our computations we will also adopt the following abuses of notation: if $k > j$ we will write u_j (a class on X_k) instead of $\pi_{j,k}^* u_j$ and similarly $s_{j,i}$ instead of $\pi_{j,k}^* s_{j,i}$. This should not lead to any confusion.

Now, from the horizontal exact sequences in the diagram page 29, the relative Euler exact sequence, Whitney's formula and formula 2.1 one easily derives (as in [Mou09]) the recursion formula

$$s_{k,\ell} = \sum_{j=0}^{\ell} M_{\ell,j}^n s_{k-1,j} u_k^{\ell-j}, \tag{2.2}$$

where $M_{\ell,j}^n = \sum_{i=0}^{\ell-j} (-1)^i \binom{n-2+i+j}{i}$; in particular $M_{\ell,\ell}^n = 1$.

Lemma 2.1.1. *Let $k \geqslant 0$, $a \geqslant 0$, $\ell \geqslant 0$, take ℓ positive integers $i_1, \dots, i_\ell$ and m divisor classes $\gamma_1, \dots, \gamma_m \in \mathcal{C}_k(X, H)$ such that $i_1 + \cdots + i_\ell + m + a = n_k$. Let $\gamma_q =: \alpha_{q,0} h + \sum_i \alpha_{q,i} u_i$. Then*

$$\int_{X_k} s_{k,i_1}\cdots s_{k,i_\ell}\gamma_1\cdots\gamma_m h^a = \sum_{j_1,\dots,j_{k+\ell},b} Q_{j_1,\dots,j_{k+\ell},b}\int_X s_{j_1}\cdots s_{j_{k+\ell}}h^{a+b},$$

where in each term of the sum we have $b \geqslant 0$ and the $Q_{j_1,\dots,j_{k+\ell},b}$ are polynomials in the $\alpha_{q,i}$ whose coefficients are independent of X. Moreover, up to reordering of the j_p one has $j_1 \leqslant i_1,\dots,j_\ell \leqslant i_\ell$.

Proof. This is an immediate induction on k. The result is clear for $k=0$. Now suppose it is true for some $k>0$ and take m divisors, $\gamma_1,\dots,\gamma_m \in \mathcal{C}_{k+1}(X)$ on X_{k+1}. Let $\gamma_q := \alpha_{q,0}h + \sum_i \alpha_{q,i}u_i$. Then using recursion formula (2.2) and expanding, we get

$$\begin{aligned}\int_{X_{k+1}} s_{k+1,i_1}\cdots s_{k+1,i_\ell}\gamma_1\cdots\gamma_m h^a &= \sum P^{p_1,\dots,p_{k+1}}_{j_i,\dots,j_\ell,b}\int_{X_{k+1}} s_{k,j_1}\cdots s_{k,j_\ell}u_{k+1}^{p_{k+1}}\cdots u_1^{p_1}h^{a+b}\\ &= \sum P^{p_1,\dots,p_{k+1}}_{j_i,\dots,j_\ell,b}\int_{X_k} s_{k,j_1}\cdots s_{k,j_\ell}s_{k,r}u_k^{p_k}\cdots u_1^{p_1}h^{a+b},\end{aligned}$$

where in each term of the sum, $r = p_{k+1}-(n-1)$ and moreover, thanks to formula 2.2 one has $j_1 \leqslant i_1,\dots,j_\ell \leqslant i_\ell$. Note also that the $P^{p_1,\dots,p_{k+1}}_{j_i,\dots,j_\ell,b}$ are polynomials in the $\alpha_{i,j}$ but their coefficients do not depend on X. Now we can conclude using the induction hypothesis. $\square$

2.2 Segre classes for complete intersections

Let us introduce the setting in which we will work for the rest of this section. Fix M a smooth N-dimensional projective variety, and H an ample line bundle on M. Let $c \geqslant 1$ and take c ample line bundles $A_1,\dots,A_c$ on M. Take $X := H_1\cap\cdots\cap H_c$, a smooth complete intersection variety, where $H_i \in |d_iA_i|$ for some $d_i \in \mathbb{N}$. We will set $n := \dim X = N-c$. Also let $h := c_1(H)$ and $\alpha_i := c_1(A_1)$ for $1 \leqslant i \leqslant c$. Moreover, let $\kappa := \lceil \frac{n}{c}\rceil$ and take b such that $n = (\kappa-1)c+b$; observe that $0 < b \leqslant c$. To simplify our formulas we also set $\hat{\imath} := i+n-1$ so that $\pi_{k-1,k_*}u_k^{\hat{\imath}} = s_{k-1,i}$.

As we will be interested in the asymptotic behavior of polynomials in $\mathbb{Z}[d_1,\dots,d_c]$ we need some more notation. Let $P \in \mathbb{Z}[d_1,\dots,d_c]$; then $\deg P$ denotes the total degree of the polynomial and P^{dom} the homogeneous part of P of degree $\deg P$. We will write $P = o(d^k)$ if $\deg P < k$ and if $Q \in \mathbb{Z}[d_1,\dots,d_c]$ is another polynomial we will write $P \sim Q$ if $P^{\mathrm{dom}} = Q^{\mathrm{dom}}$, and $P \gtrsim Q$ if $P^{\mathrm{dom}} \geqslant Q^{\mathrm{dom}}$ (i.e., $\forall (d_1,\dots,d_c) \in \mathbb{N}^c$, $P^{\mathrm{dom}}(d_1,\dots,d_c) \geqslant Q^{\mathrm{dom}}(d_1,\dots,d_c)$).

Moreover, some of our computations will take place in the polynomial ring $A^*(X)[d_1,\dots,d_c]$ where $A^*(X)$ denotes the Chow ring of X. The notation we will use are the natural ones. If $P \in A^\ell(X)[d_1,\dots,d_c]$, we will write $\deg P$ for the degree of P as a polynomial in the d_i, and P^{dom} for the homogenous part of degree $\deg P$ of P. First we compute the Segre classes of Ω_X. To fix notation, let

$$s(\Omega_M) = 1+\tau_1+\cdots+\tau_N,$$

where $\tau_i \in A^i(M)$. Now, the conormal bundle exact sequence

$$0 \to \bigoplus_{i=1}^c A_{i_{|X}}^{-d_i} \to \Omega_{M_{|X}} \to \Omega_X \to 0$$

yields

$$\begin{aligned} s(\Omega_X) &= \frac{s(\Omega_{M_{|X}})}{s(\bigoplus_{i=1}^c A_{i_{|X}}^{-d_i})} = s(\Omega_{M_{|X}})c\left(\bigoplus_{i=1}^c A_{i_{|X}}^{d_i}\right)\\ &= (1+\tau_{1_{|X}}+\cdots+\tau_{n_{|X}})\prod_{i=1}^c(1+d_i\alpha_{i_{|X}}).\end{aligned}$$

Expanding the right-hand side as a polynomial in $A^*(X)[d_1,\dots,d_c]$, we see that $\deg(s_\ell)=\min\{\ell,c\}$. When $\ell\leqslant c$ one has moreover

$$s_\ell^{\mathrm{dom}}(\Omega_X)=\sum_{j_1<\cdots<j_\ell} d_{j_1}\cdots d_{j_\ell}\alpha_{j_1|X}\cdots\alpha_{j_\ell|X}=c_\ell^{\mathrm{dom}}\left(\bigoplus_{i=1}^{c}A_{i|X}^{d_i}\right). \tag{2.3}$$

Remark 2.2.1. If $n\leqslant c$, equality (2.3) holds for all $\ell\in\mathbb{Z}$ (the case $\ell>n$ is obvious since both side of the equality vanish by a dimension argument).

Remark 2.2.2. The important point is that the α_i and the τ_i are independent of X. So that in particular the intersection products involving the α_i and the τ_i are independent of X as well. This in turn implies that the intersection products involving the s_i depend only the d_i and of the intersections of the α_i and the τ_i.

With this we can give estimates for some intersection products on X.

Lemma 2.2.3. a) *Let $0\leqslant i_1\leqslant\cdots\leqslant i_k$, $\ell>0$ such that $i_1+\cdots+i_k+\ell=n$. Then*

$$\deg\left(\int_X s_{i_1}\cdots s_{i_k}h^\ell\right)<N.$$

b) *Let $0\leqslant i_1\leqslant\cdots\leqslant i_k$ such that $i_1+\cdots+i_k=n$. Then $\int_X s_{i_1}\cdots s_{i_k}$ is of degree N if and only if $i_k\leqslant c$.*

c) *Let $0\leqslant i_1\leqslant\cdots\leqslant i_\kappa$. If $i_1<b$ or if $i_1=b$ and $i_j<c$ for some $j>1$, then*

$$\deg\left(\int_X s_{i_1}\cdots s_{i_\kappa}\right)<N.$$

Proof. Let $0\leqslant\ell\leqslant n$. Observe that

$$\int_X s_{i_1}\cdots s_{i_k}h^\ell=d_1\cdots d_c\int_M \alpha_1\cdots\alpha_c\cdot s_{i_1}\cdots s_{i_k}h^\ell.$$

Therefore,

$$\deg\left(\int_X s_{i_1}\cdots s_{i_k}h^\ell\right)=\deg s_{i_1}+\cdots+\deg s_{i_k}+c.$$

Recall also that $\deg(s_i)=\min\{i,c\}$. Now, for the first point, let $\ell>0$. Then,

$$\deg\left(\int_X s_{i_1}\cdots s_{i_k}h^\ell\right)\quad\leqslant\quad i_1+\cdots+i_k+c<n+c=N.$$

To see the second point, just observe that

$$\deg\left(\int_X s_{i_1}\cdots s_{i_k}\right)=\sum_{j=1}^{k}\min\{i_j,c\}\leqslant\sum_{j=1}^{k}i_j=N$$

and equality holds if and only if $i_j\leqslant c$ for all $1\leqslant j\leqslant k$. The last point is an easy consequence of the second one thanks to the equality $n=(\kappa-1)c+b$. $\square$

2.3 Segre classes for complete intersections in $\mathbb{P}^N$

We will need to have a more precise statement in the case of complete intersection in $\mathbb{P}^N$. Let us introduce our notation. In this section, we will take $d_1,\dots,d_c\in\mathbb{N}$, for each $1\leqslant i\leqslant c$ we take $\sigma_i\in H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(d_i))$ such that $X:=H_1\cap\cdots\cap H_c$ is a smooth complete intersection, where $H_i:=(\sigma_i=0)$. We also set $h_{\mathbb{P}^N}:=c_1(\mathcal{O}_{\mathbb{P}^N}(1))$ and $h:=c_1(\mathcal{O}_X(1))$. If P is a polynomial in $\mathbb{Z}[d_1,\dots,d_c,h]$ homogeneous in h and of degree k in h, we will write $\tilde{P}$ for the unique polynomial in $\mathbb{Z}[d_1,\dots,d_c]$ satisfying $P=\tilde{P}h^k$.

Let us detail our computations. Let $m \in \mathbb{Z}$. The twisted Euler exact sequence

$$0 \to \Omega_{\mathbb{P}^N}(-m) \to \mathcal{O}_{\mathbb{P}^N}^{\oplus N+1}(-m-1) \to \mathcal{O}_{\mathbb{P}^N}(-m) \to 0$$

yields:

$$s(\Omega_{\mathbb{P}^N}(-m)) = \frac{s(\mathcal{O}_{\mathbb{P}^N}^{\oplus N+1}(-m-1))}{s(\mathcal{O}_{\mathbb{P}^N}(-m))} = \frac{(1+mh_{\mathbb{P}^N})}{(1+(1+m)h_{\mathbb{P}^N})^{N+1}}.$$

The (twisted) conormal bundle exact sequence

$$0 \to \bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i - m) \to \Omega_{\mathbb{P}^N_{|X}}(-m) \to \Omega_X(-m) \to 0$$

yields

$$\begin{aligned} s(\Omega_X(-m)) &= \frac{s(\Omega_{\mathbb{P}^N_{|X}}(-m))}{s(\bigoplus_{i=1}^c \mathcal{O}_X(-d_i-m))} = \frac{(1+mh)\prod_{i=1}^c(1+(d_i+m)h)}{(1+(1+m)h)^{N+1}} \\ &= (1-(1+m)h+(1+m)^2h^2-\cdots)^{N+1}(1+mh)\prod_{i=1}^{c}(1+(d_i+m)h). \end{aligned}$$

Expanding the right-hand side as a polynomial in $\mathbb{Z}[d_1,\dots,d_c,h]$, we see that for $\ell \geqslant c$ we have $\deg(\tilde{s_\ell}) = c$ and that for $\ell \leqslant c$ we have

$$s_\ell^{\mathrm{dom}}(\Omega_X(-m)) = \sum_{j_1<\cdots<j_\ell} d_{j_1}\cdots d_{j_\ell}h^\ell = c_\ell^{\mathrm{dom}}\left(\bigoplus_{i=1}^{c}\mathcal{O}_X(d_i+m)\right) = c_\ell\left(\bigoplus_{i=1}^{c}\mathcal{O}_X(d_i)\right). \tag{2.4}$$

We also need an even more precise statement in the case $m = 0$, because we want to have an explicit bound to some of our results, and to achieve this one has to know exactly what are the lower order terms. Let us introduce another notation: for $0 \leqslant i \leqslant c$, let

$$\epsilon_i(d_1,\dots,d_c) := \sum_{1\leqslant j_1<\cdots<j_i\leqslant c} d_{j_1}\cdots d_{j_i}$$

and, $\epsilon_i(d_1,\dots,d_c) = 0$ if $i > c$. Now, by expending and since $h^k = 0$ if $k > n$, we get

$$\begin{aligned} s(\Omega_X) &= (1-h+h^2-\cdots)^{N+1}\prod_{i=1}^{c}(1+d_ih) = \left(\sum_{k=0}^{n}\binom{N+k}{k}(-1)^kh^k\right)\left(\sum_{i=0}^{c}\epsilon_i(d_1,\dots,d_c)h^i\right) \\ &= \sum_{k=0}^{n}\sum_{i=0}^{n}\binom{N+k}{k}(-1)^k\epsilon_i(d_1,\dots,d_c)h^{i+k} = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{N+k}{k}(-1)^k\epsilon_{j-k}(d_1,\dots,d_c)h^j. \end{aligned}$$

From this we deduce the explicit form of the Segre classes that will be used afterwards:

$$s_j(\Omega_X) = \sum_{k=0}^{j}\binom{N+k}{N}(-1)^k\epsilon_{j-k}(d_1,\dots,d_c)h^j. \tag{2.5}$$

Chapter 3

Numerical positivity of the cotangent bundle

In this chapter we prove that the cotangent bundle of a complete intersection variety of high multidegree is numerically positive. Based on the computations of Chapter 2 we give a simple argument that will give the general result. However, this gives no bound on the degree. We make a more thorough treatment of those computations in dimension 2.

3.1 General result

The main result of this chapter is the following.

Theorem 3.1.1. *Fix $a \in \mathbb{Z}$. There exists $D_{N,n,a} \in \mathbb{N}$ such that if $X \subset \mathbb{P}^N$ is a complete intersection of dimension n, of codimension c and multidegree $(d_1, \ldots, d_c)$ such that $c \geqslant n$ and $d_i > D_{N,n,a}$ for all i then $\Omega_X(-a)$ is numerically positive.*

Proof. By Lemma 1.2.7 we have to check that for any subvariety $Y \subseteq X$ of dimension ℓ and for any partition λ of ℓ one has $\int_Y \Delta_{\bar{\lambda}}(s(\Omega_X(-a))) > 0$. Moreover, $\int_Y \Delta_{\bar{\lambda}}(s(\Omega_X(-a))) = \tilde{\Delta}_{\bar{\lambda}}(s(\Omega_X(-a))) \int_Y h^{\ell}$, thus we just have to check that $\tilde{\Delta}_{\bar{\lambda}}(s(\Omega_X(-a))) > 0$ when the d_i are large enough, which is equivalent to $\tilde{\Delta}^{\mathrm{dom}}_{\bar{\lambda}}(s(\Omega_X(-a))) > 0$. Now the equality

$$\tilde{\Delta}^{\mathrm{dom}}_{\bar{\lambda}}(s(\Omega_X(-a))) = \det(\tilde{s}^{\mathrm{dom}}_{\lambda_i+j-i}(\Omega_X(-a)))_{1 \leqslant i,j \leqslant \ell} \tag{3.1}$$

holds if one can prove that the right-hand side is non zero. But by (2.4) we find:

$$\begin{aligned} \det(\tilde{s}^{\mathrm{dom}}_{\lambda_i+j-i}(\Omega_X(-a)))_{1 \leqslant i,j \leqslant l} &= \det\left(\tilde{c}_{\lambda_i+j-i}\left(\bigoplus_{j=1}^{k} \mathcal{O}(d_j)\right)\right)_{1 \leqslant i,j \leqslant l} \\ &= \tilde{\Delta}_{\bar{\lambda}}\left(c\left(\bigoplus_{j=1}^{k} \mathcal{O}(d_j)\right)\right). \end{aligned}$$

By applying the theorem of Fulton and Lazarsfeld to $\bigoplus_{j=1}^{k} \mathcal{O}_X(d_j)$ (which is ample if $d_i > 0$) we find that this is positive. This yields equality in (3.1), and we get the desired result. $\square$

Remark 3.1.2. Note that there are no assumptions made on the genericity of our complete intersection.

This argument hides all the combinatorics that appear if one wants to make the computations explicit.

3.2 Surfaces in $\mathbb{P}^N$

In the case of surfaces one can make all the computations explicit without too much difficulty. Therefore one can obtain an effective bound in this situation.

Take $N \geqslant 4$ and $N-2$ hypersurfaces $H_1, \dots, H_{N-2}$ with $d_i := \deg(H_i) \geqslant 2$. Using the formulas from Chapter 2 we obtain by a quick computation

$$\begin{aligned}
\Delta_{(1,0)}(\Omega_X) &= s_1(\Omega_X) = \sum_i d_i - (N+1), \\
\Delta_{(1,1)}(\Omega_X) &= s_2(\Omega_X) = \sum_{i \leqslant j} d_i d_j - (N+1)\sum_i d_i + \binom{N+2}{2}, \\
\Delta_{(2,0)}(\Omega_X) &= c_2(\Omega_X) = \sum_{i<j} d_i d_j - (N+1)\sum_i d_i + \binom{N+1}{2}.
\end{aligned}$$

Observe that

$$\Delta_{(1,1)}(\Omega_X) = \Delta_{(2,0)}(\Omega_X) + \sum_i d_i^2 + (N+1).$$

Therefore, we obtain

Proposition 3.2.1. *With those hypothesis, Ω_X is numerically positive if and only if $c_2 > 0$.*

From this we can easily derive a bound. Let $d_1 = \cdots = d_{N-2} = x$. We want to know when

$$\binom{N-2}{2} x^2 - (N+1)(N-2)x + \binom{N+1}{2} > 0. \tag{3.2}$$

Once we solved this, we let

$$\delta(N) := \left\lceil \frac{(N+1)(N-2) + \sqrt{2(N+1)(N-1)(N-2)}}{(N-2)(N-3)} \right\rceil,$$

chosen such that x satisfies (3.2) as soon as $x > \delta_N$. Up to a little check, we see that if for all $1 \leqslant i \leqslant N-2$, we have $d_i \geqslant N-2$, then Ω_X is numerically positive. This bound is optimal. We can write it even more explicitly:

$$\begin{aligned}
\delta(4) &= 9, \\
\delta(5) &= 5, \\
\delta(N) &= 4 \quad \text{for} \quad N = 6, 7, \\
\delta(N) &= 3 \quad \text{for} \quad 8 \leqslant N \leqslant 12, \\
\delta(N) &= 2 \quad \text{for} \quad 13 \leqslant N.
\end{aligned}$$

Chapter 4

Jet differentials on complete intersection varieties

This chapter is motivated by two theorems of Diverio (see [Div08] and [Div09]). In [Div09], he proved a nonvaninshing theorem for jet differentials on hypersurfaces of high degree.

Theorem ([Div09] Theorem 1)**.** *Fix $n \geqslant 1$, fix $k \geqslant n$ and fix $a > 0$. There exists an integer $d_{n,k}$ such that, if $X \subset \mathbb{P}^{n+1}$ is a smooth projective hypersurface of degree greater than $d_{n,k}$ and if $m \in \mathbb{N}$ is big enough, then*

$$H^0(X, E_{k,m}\Omega_X \otimes \mathcal{O}_X(-a)) \neq 0.$$

Using a theorem of Bruckman-Rackwitz [BR90], Diverio also proved a vanishing theorem for jet differentials on complete intersection varieties.

Theorem ([Div08] Theorem 7)**.** *Let X be a complete intersection variety in $\mathbb{P}^N$, of dimension n and codimension c. For all $m \geqslant 1$ and $1 \leqslant k < \lceil \frac{n}{c} \rceil$, one has*

$$H^0(X, E_{k,m}\Omega_X) = 0.$$

It seems therefore natural to look for the non-vanishing of $H^0(X, E_{\lceil \frac{n}{c} \rceil, m}\Omega_X)$ when X is a smooth complete intersection of dimension n and codimension c of high multidegree. This is the content of Theorem 4.1.1.

4.1 Non-vanishing for jet differentials

We use the notation of Chapter 2. We can now state and prove our non-vanishing theorem; it will show that Diverio's result ([Div08] Theorem 7) is optimal in k.

Theorem 4.1.1. *Let M be a N-dimensional projective variety and H an ample divisor on M. Fix $a \in \mathbb{N}$, fix $1 \leqslant c \leqslant N-1$ and $k \geqslant \kappa := \lceil \frac{n}{c} \rceil$. Take $A_1, \dots, A_c$ ample line bundles on M. For $d_1, \dots, d_c \in \mathbb{N}$ big enough, take generic hypersurfaces $H_1 \in |d_1 A_1|, \dots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \dots \cap H_c$. Then $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ is big on X_k. In particular, when $m \gg 0$,*

$$H^0(X, E_{k,m}\Omega_X \otimes H^{-ma}) \neq 0.$$

Remark 4.1.2. We will give an explicit bound on $\Gamma_{N,n,a}$ when $M = \mathbb{P}^N$ and $\kappa = 1$ in section 4.2 .

Take $r_0 \in \mathbb{N}$ such that $\Omega_M \otimes H^{r_0}$ is nef. Then, as in [Div08] and [Dem97a] one can show that the line bundle

$$L_k := \mathcal{O}_{X_k}(2 \cdot 3^{k-2}, \dots, 2 \cdot 3, 2, 1) \otimes \pi_k^* H^{r_0 \cdot 3^{k-1}}$$

is nef.

Remark 4.1.3. Diverio ([Div08] Lemma 3) initially proved that L_k is nef in the case $M = \mathbb{P}^N$, $X \subset \mathbb{P}^N$ a hypersurface and $H = \mathcal{O}_{\mathbb{P}^N_{|X}}(1)$, in which case $r_0 = 2$ is sufficient. But his proof works just as well in this more general setting. Moreover X could be any smooth subvariety of M, we don't need the complete intersection hypothesis for this.

We can write the first Chern class of L_k,

$$\ell_k := c_1(L_k) = u_k + \beta_k,$$

where β_k is a class that comes from X_{k-1}. Now we can state the main technical Lemma. This is just the combination of Lemmas 2.1.1 and 2.2.3.

Lemma 4.1.4. *With the above notation we have the following estimates.*

a) *Let $k \geqslant 1$ and $\gamma_1, \ldots, \gamma_{n_k-1} \in \mathcal{C}_k(X,H)$. Then*

$$\int_{X_k} \gamma_1 \cdots \gamma_m h = o(d^N).$$

b) *Let $\gamma_1, \ldots, \gamma_p \in \mathcal{C}_k(X,H)$ and $0 \leqslant i_1 \leqslant \cdots \leqslant i_q$ such that $p + \sum i_j = n_k$. If $i_1 < b$, or if $i_1 = b$ and $i_j < c$ for some $j > 1$, then*

$$\int_{X_k} s_{k,i_1} \cdots s_{k,i_q} \gamma_1 \cdots \gamma_p = o(d^N), \tag{4.1}$$

$$\int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q} \gamma_1 \cdots \gamma_p = o(d^N). \tag{4.2}$$

c) *Let $0 < k < \kappa$. Then*

$$\int_{X_k} s_{k,b} s_{k,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} = \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-k} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N).$$

Proof. a) is an immediate consequence of Lemma 2.2.3.a) and Lemma 2.1.1. Similarly for b): thanks to Lemma 2.1.1, we write

$$\int_{X_k} s_{k,i_1} \cdots s_{k,i_q} \gamma_1 \cdots \gamma_p = \sum_{j_1,\ldots,j_{k+q}} Q_{j_1,\ldots,j_{k+q}} \int_X s_{j_1} \cdots s_{j_{k+q}} h^a,$$

where $a \geqslant 0$ and moreover we know that, because $j_s \leqslant i_s$ for all s, in each term of this sum either $j_1 < b$ or $j_p < c$ for some $p > 0$. Thus we can apply Lemma 2.2.3.a) (if $a > 0$) or 2.2.3.b) (if $a = 0$). From this, one can easily deduce formula (4.2): write $\gamma_i = a_i u_i + \beta_i$, where $\beta_i \in \mathcal{C}_{k-1}(X)$. Then,

$$\begin{aligned}
\int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q}\ \gamma_1 \cdots \gamma_p &= \int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q} (a_1 u_1 + \beta_1) \cdots (a_1 u_1 + \beta_p) \\
&= \sum_{I \subseteq \{1,\cdots,p\}} \left(\prod_{i \in I} a_i \right) \int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q} u_k^{|I|} \prod_{i \notin I} \beta_i \\
&= \sum_{I \subseteq \{1,\cdots,p\}} \left(\prod_{i \in I} a_i \right) \int_{X_{k-1}} s_{k-1,i_1} \cdots s_{k-1,i_q} \cdot s_{k-1,|I|-n+1} \prod_{i \notin I} \beta_i
\end{aligned}$$

and we conclude by applying formula (4.1).

To see c), we have

$$\begin{aligned}\int_{X_k} s_{k,b} s_{k,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} &= \int_{X_k} \left(\sum_{i=0}^{b} M_{b,i}^n s_{k-1,i} u_k^{b-i} \right) \left(\sum_{i=0}^{c} M_{c,i}^n s_{k-1,i} u_k^{c-i} \right)^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} \\ &= \int_{X_k} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N)\end{aligned}$$

obtained by expanding and using formula (4.2) in each term of the obtained sum. Now,

$$\begin{aligned}\int_{X_k} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N) &= \int_{X_k} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} (u_k + \beta_k)^{\hat{c}} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} \\ &= \int_{X_k} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} \sum_{i=0}^{\hat{c}} \binom{\hat{c}}{i} u_k^{\hat{c}-i} \beta_k^i \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} \\ &= \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} \sum_{i=0}^{c} \binom{\hat{c}}{i} s_{k-1,i} \beta_k^i \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} \\ &= \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-k-1} s_{k-1,c} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N) \\ &= \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-(k-1)-1} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N).\end{aligned}$$

□

Recall also the following consequence of Demailly's holomorphic Morse inequalities (see for example [Laz04a]).

Theorem 4.1.5. *Let Y be a smooth projective variety of dimension n and let F and G be nef divisors on Y. If $F^n > nG \cdot F^{n-1}$, then $F - G$ is big.*

We are now ready to prove Theorem 4.1.1.

Proof. First we recall an argument from [Div09] to show that we just have to check that $\mathcal{O}_{X_k}(a_1, \cdots, a_k) \otimes \pi_k^* \mathcal{O}_X(-a)$ is big for some suitable a_i. We know (see [Dem97a]) that $D_k := \mathbb{P}(\Omega_{X_{k-1}/X_{k-2}}) \subset X_k$ is an effective divisor which satisfies the relation $\pi_{k-1,k}^* \mathcal{O}_{X_{k-1}}(1) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-D_k)$. From this, an immediate induction shows that for any $k > 1$ and any k-uple $(a_1, \dots, a_k)$ we have

$$\mathcal{O}_{X_k}(b_{k+1}) = \mathcal{O}_{X_k}(a_1, \dots, a_k) \otimes \pi_{2,k}^* \mathcal{O}_{X_2}(b_1 D_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(b_{k-1} D_k),$$

where for all $j > 0$, $b_j := a_1 + \cdots + a_j$. Thus when $0 \leqslant b_j$ for all $0 \leqslant j \leqslant k$ then $\pi_{2,k}^* \mathcal{O}_{X_2}(b_1 D_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(b_{k-1} D_k)$ is effective, this means that, under this condition, to prove that $\mathcal{O}_{X_k}(1) \otimes \pi_k^* \mathcal{O}_X(-a)$ is big it is sufficient to show that $\mathcal{O}_{X_k}(a_1, \dots, a_k) \otimes \pi_k^* \mathcal{O}_X(-a)$ is big.

Let $D = F - G$ where, as in [Mou09], we set $F := L_k \otimes \cdots \otimes L_1$, and $G = \pi_k^* H^{m+a}$ where $m \geqslant 0$ is chosen so that $F \otimes \pi_k^* H^{-m}$ has no component coming from X. It is therefore sufficent to show that D is big. To do so, we will apply holomorphic Morse inequalities to F and G (both nef). We need to prove that

$$F^{n_k} > n_k F^{n_k - 1} \cdot G.$$

Clearly, the right-hand side has degree strictly less than N in the d_i thanks to Lemma 4.1.4 and therefore we just have to show that the left hand side is larger than a positive polynomial of degree N in the d_i. Let $\alpha := c_1(\pi_k^*(\mathcal{O}_X(a)),$

$$
\begin{aligned}
F^{n_k} &= \int_{X_k} (\ell_k + \cdots + \ell_1 - \alpha)^{n_k} \\
&= \int_{X_k} \sum_{i=0}^{n_k} (-1)^i \binom{n_k}{i} (\ell_k + \cdots + \ell_1)^{n_k - i} \alpha^i \\
&= \int_{X_k} (\ell_k + \cdots + \ell_1)^{n_k} + o(d^N)
\end{aligned}
$$

by applying Lemma 4.1.4. But since all the ℓ_i are nef,

$$
\begin{aligned}
\int_{X_k} (\ell_l + \cdots + \ell_1)^{n_k} &\geqslant \int_{X_k} \ell_k^{n-1} \cdots \ell_{\kappa+1}^{n-1} \cdot \ell_\kappa^{\hat{b}} \cdot \ell_{\kappa-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} \\
&= \int_{X_\kappa} \ell_\kappa^{\hat{b}} \cdot \ell_{\kappa-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} \\
&= \int_{X_{\kappa-1}} s_{\kappa-1,b} \cdot \ell_{\kappa-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N)
\end{aligned}
$$

The last inequality is obtained by using Lemma 4.1.4.2. Now an immediate induction proves that for all $k < \kappa$ one has

$$
F^{n_\kappa} \geqslant \int_{X_k} s_{k,b} s_{k,c}^{\kappa-k-1} l_k^{\hat{c}} \cdots l_1^{\hat{c}} + o(d^N).
$$

We just proved the case $k = \kappa - 1$ and the other part of the induction is exactly the content of Lemma 4.1.4.3. Therefore,

$$
F^{n_\kappa} \geqslant \int_X s_b s_c^{\kappa-1} + o(d^N).
$$

Applying Lemma 2.2.3.2, we see that $\deg \int_X s_b s_c^{\kappa-1} = N$, thus

$$
F^{n_\kappa} \geqslant \int_X \left(s_b s_c^{\kappa-1}\right)^{dom} + o(d^N).
$$

We just have to check that $\int_X \left(s_b s_c^{\kappa-1}\right)^{dom} > 0$. But this follows from formula (2.3) and the fact that all the α_i are ample (recall that $\alpha_i = c_1(A_i)$). $\square$

4.2 Effective existence of symmetric differentials

The special case when in Theorem 4.1.1, $M := \mathbb{P}^N$, $A_1, \ldots, A_c = H = \mathcal{O}_{\mathbb{P}^N}(1)$ and $\kappa = 1$ (that is, when the codimension is greater than the dimension) is of particular interest to us here, therefore we will give an effective bound on the degree in this case. This allows us to give an explicit bound on the degree in our main results. First we rewrite this theorem in the present situation.

Theorem 4.2.1. *Fix $a \in \mathbb{N}$. Then there exists a constant $\Gamma_{N,n,a}$ such that if $X \subset \mathbb{P}^N$ is a smooth complete intersection of dimension n, codimension c and multidegree $(d_1, \ldots, d_c)$ satisfying $n \leqslant c$ and $d_i \geqslant \Gamma_{N,n,a}$ then $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi_{\Omega_X}^* \mathcal{O}_X(-a)$ is big. In particular, when $m \gg 0$,*

$$
H^0(X, S^m \Omega_X \otimes \mathcal{O}_X(-am)) \neq 0.
$$

We give a rough bound on $\Gamma_{N,n,a}$ that works for any N, n, a and afterwards we give a better bound when $n = 2$.

Remark 4.2.2. We would like to mention that O. Debarre proved in [Deb05], using Riemann-Roch computations, that $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi_{\Omega_X}^* \mathcal{O}_X(1)$ is big under the assumptions on the dimension and the degree.

Here we do explicitly, in this context, the intersection computations we did during the proof of 4.1.1. We take the same notation, under the assumption $n \leqslant c$.

$$\begin{aligned} F^{2n-1} &- (2n-1)F^{2n-2}\cdot G \\ &= \int_{\mathbb{P}(\Omega_X)} (u+2h)^{2n-1} - (2n-1)(u+2h)^{2n-2}(2+a)h \\ &= \int_{\mathbb{P}(\Omega_X)} \sum_{i=0}^{2n-1} \binom{2n-1}{i} u^{2n-1-i}(2h)^i - (2n-1)(2+a)\sum_{j=0}^{2n-2}\binom{2n-2}{j} u^{2n-2-j}(2h)^j h \\ &= \int_{\mathbb{P}(\Omega_X)} u^{2n-1} + \sum_{i=1}^{2n-1}\left(2^i\binom{2n-1}{i} - (2n-1)(2+a)2^{i-1}\binom{2n-2}{i-1}\right) u^{2n-1-i}h^i \\ &= \int_{\mathbb{P}(\Omega_X)} u^{2n-1} + \sum_{i=1}^{2n-1} 2^{i-1}(2-i(2+a))\binom{2n-1}{i} u^{2n-1-i}h^i \\ &= \int_X \sum_{i=0}^{n} 2^{i-1}(2-i(2+a))\binom{2n-1}{i} s_{n-i}h^i. \end{aligned}$$

Now we will use formula (2.5) to see how this intersection product depend on the multidegree $(d_1,\ldots,d_c)$. To ease the notation we will also just write ϵ_i instead of $\epsilon_i(d_1,\ldots,d_c)$.

$$\begin{aligned} F^{2n-1} &- (2n-1)F^{2n-2}\cdot G \\ &= \sum_{i=0}^{n}\int_X 2^{i-1}(2-i(2+a))\binom{2n-1}{i} s_{n-i}h^i \\ &= \sum_{i=0}^{n} 2^{i-1}(2-i(2+a))\binom{2n-1}{i}\left(\sum_{k=0}^{n-i}\binom{N+k}{N}(-1)^k\epsilon_{n-i-k}\right) \\ &= \sum_{i=0}^{n}\sum_{k=0}^{n-i}(-1)^k 2^{i-1}(2-i(2+a))\binom{2n-1}{i}\binom{N+k}{N}\epsilon_{n-i-k} \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-j}(-1)^{n-i-j}2^{i-1}(2-i(2+a))\binom{2n-1}{i}\binom{N+n-i-j}{N}\epsilon_j \\ &= \sum_{j=0}^{n} \mathrm{D}_a^{N,n,j}\,\epsilon_j(d_1,\ldots,d_c), \end{aligned}$$

where

$$\mathrm{D}_a^{N,n,j} = (-1)^{n-j}\sum_{i=0}^{n-j}(-1)^i 2^{i-1}(2-i(2+a))\binom{2n-1}{i}\binom{N+n-i-j}{N}.$$

Rough Bound

Now we can give a straightforward rough bound for $\Gamma_{N,n,a}$ for any N,n,a. Let us recall a basic fact to estimate the zero locus of a polynomial in one real variable.

Lemma 4.2.3. *Let* $P(x) := x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$. *If* $x \geqslant 1 + \max_i |a_i|$ *then* $P(x) > 0$.

Now we will see how to apply this in our situation. As previously, we write

$$\epsilon_i(x_1,\ldots,x_c) := \sum_{1\leqslant j_1<\cdots<j_i\leqslant c} x_{j_1}\cdots x_{j_i}.$$

Now take $k \leqslant c$ and

$$P(x_1,\dots,x_c) = \epsilon_k(x_1,\dots,x_c) + a_{k-1}\epsilon_{k-1}(x_1,\dots,x_c) + \cdots + a_1\epsilon_1(x_1,\dots,x_c) + a_0.$$

We want to find $r \in \mathbb{R}$ such that if $x_i \geqslant r$ for all $1 \leqslant i \leqslant c$ then $P(x_1,\dots,x_c) > 0$. We will be done if we are able to find $r \in \mathbb{R}$ satisfying $P(r,\dots,r) > 0$ and $\frac{\partial P}{\partial x_i}(x_1,\dots,x_c) > 0$ for all $1 \leqslant i \leqslant c$ as soon as $x_j \geqslant r$ for all $1 \leqslant j \leqslant c$. Now observe that since the ϵ_i are symmetric, we have to check the positivity of just one partial derivative, say $\frac{\partial P}{\partial x_c}$. By induction, we are left to find $r \in \mathbb{R}$ satisfying $P(r,\dots,r) > 0$, $\frac{\partial P}{\partial x_c}(r,\dots,r) > 0$, $\frac{\partial}{\partial x_{c-1}}\frac{\partial P}{\partial x_c}(r,\dots,r) > 0$, $\dots$, $\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_{c-1}}\frac{\partial P}{\partial x_c}(r) > 0$. But we also have

$$\frac{\partial \epsilon_j}{\partial x_c}(x_1,\dots,x_c) = \epsilon_{j-1}(x_1,\dots,x_{c-1}),$$

and thus

$$\frac{\partial P}{\partial x_c} = \frac{\partial}{\partial x_c}\sum_{i=0}^{k} a_i\epsilon_i(x_1,\dots,x_c) = \sum_{i=0}^{k-1} a_{i+1}\epsilon_i(x_1,\dots,x_{c-1}),$$

where $a_k = 1$. Similarly

$$\frac{\partial}{\partial x_{c-j}}\cdots\frac{\partial}{\partial x_{c-1}}\frac{\partial P}{\partial x_c} = \sum_{i=0}^{k-j} a_{i+j}\epsilon_i(x_1,\dots,x_{c-j}).$$

Evaluating in $(r,\dots,r)$ yields, for any $0 \leqslant j \leqslant c$,

$$\frac{\partial}{\partial x_{c-j}}\cdots\frac{\partial}{\partial x_{c-1}}\frac{\partial P}{\partial x_c}(r,\dots,r) = \sum_{i=0}^{k-j} a_{i+j}\binom{c-j}{i} r^i.$$

Now we apply Lemma 4.2.3 to each of those polynomials and therefore we just have to give a bound for

$$\max_{\substack{0\leqslant j\leqslant k-1\\ j\leqslant i\leqslant k-1}} \left| a_i \frac{\binom{c-j}{i-j}}{\binom{c-j}{k-j}} \right|.$$

Since

$$\frac{\binom{c-j}{i-j}}{\binom{c-j}{k-j}} \leqslant \frac{\binom{c}{i}}{\binom{c}{k}},$$

we just have to give a bound for

$$\max_{0\leqslant i\leqslant k-1} \left| a_i \frac{\binom{c}{i}}{\binom{c}{k}} \right|.$$

This is what we will do now in our intersection product computation, that is when $k = n$, $c = N - n$ and $a_j = \mathrm{D}_a^{N,n,j}$.

$$\begin{aligned}
\left|\mathrm{D}_a^{N,n,j}\,\frac{\binom{c}{j}}{\binom{c}{n}}\right| &= \left|\sum_{i=0}^{n-j}(-1)^{n-i-j}2^{i-1}(2-i(2+a))\binom{2n-1}{i}\binom{N+n-i-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}}\right| \\
&\leqslant \binom{N+n-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}} + \sum_{i=1}^{n-j}2^{i-1}(i(2+a)-2)\binom{2n-1}{i}\binom{N+n-i-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}} \\
&\leqslant \binom{N+n-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}} + 2^{n-j-1}((n-j)(2+a)-2)\sum_{i=1}^{n-j}\binom{2n-1}{i}\binom{N+n-i-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}} \\
&\leqslant \binom{N+n-j}{N}\frac{\binom{c}{j}}{\binom{c}{n}} + 2^{n-j-1}((n-j)(2+a)-2)(n-j)\binom{N+n-j-1}{N}\frac{\binom{c}{j}}{\binom{c}{n}}\binom{2n-1}{n-j} \\
&\leqslant \left(2^{n-1}(n(2+a)-2)\frac{n^2}{(N+1)}\binom{2n-1}{n}+1\right)\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(N+n)!(N-2n)!}{N!(N-n)!}.
\end{aligned}$$

This gives the desired (rough) bound for $\Gamma_{N,n,a}$:

$$\Gamma_{N,n,a} \leqslant \left(2^{n-1}(n(2+a)-2)\frac{n^2}{(N+1)}\binom{2n-1}{n}+1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(N+n)!(N-2n)!}{N!(N-n)!}.$$

Obviously this is far from optimal, but we won't go into more details for the general case. However even with such an estimate we can make a noteworthy remark. In our main theorem (Theorem 5.1.1) we will use $\Gamma_{N,n,a+N}$. Now if we fix n and if we let go N to infinity (that is when the codimension becomes bigger and bigger) we get

$$\lim_{N\to+\infty} \Gamma_{N,n,N+a} \leqslant 2^{n-1}n^3\binom{2n-1}{n}\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

That is to say that this bound decreases as c gets bigger, and will have a limit depending only on n. This corresponds to the intuition that as the codimension increases the situation improves and the multidegree can be taken smaller. This feature should be a part of any bound on Γ.

Bound in dimension two

Here we give a better bound in the case of surfaces. This is of particular interest to us since it is the case where we will have the strongest conclusion. Take the notation of the previous section, and let $n = 2$ so that $c = N - 2$. Fix $a \in \mathbb{N}$ (the case $a = N + 1$ is the most important). We want to estimate $F^3 - 3F^2 \cdot G = \sum_{j=0}^{2} \mathrm{D}_a^{N,2,j}\, \epsilon_j(d_1, \ldots, d_c)$, where

$$\begin{aligned}
\mathrm{D}_a^{N,2,2} &= 1,\\
\mathrm{D}_a^{N,2,1} &= -(N+1) - 3a,\\
\mathrm{D}_a^{N,2,0} &= \binom{N+2}{N} + 3a(N+1) - 12(a+1).
\end{aligned}$$

Observe that $\mathrm{D}_a^{N,2,0} \geqslant 0$ when $N \geqslant 4$, thus $F^3 - 3F^2 \cdot G \leqslant \epsilon_2(d_1, \ldots, d_c) - \mathrm{D}_a^{N,2,1}\, \epsilon_1(d_1, \ldots, d_c)$, and therefore we just have to bound one term,

$$\left|\mathrm{D}_a^{N,2,1}\, \frac{\binom{N-2}{1}}{\binom{N-2}{2}}\right| = 2\frac{N+1+3a}{N-3}.$$

Thus $\Gamma_{N,2,a} \leqslant 2\frac{N+1+3a}{N-3}$, and in particular $\delta_{N,2,1} = \Gamma_{N,2,N+1} \leqslant \frac{8N+8}{N-3}$.

Chapter 5

Almost everywhere ampleness

In this chapter we prove that under the hypotheses of Debarre's conjecture, the cotangent bundle is strongly big and almost everywhere ample. As a corollary we prove Debarre's conjecture in the case of surfaces.

5.1 Almost everywhere ampleness

We start by the main statement and its consequences before giving the proof. During the proof we need a global generation statement for some twisted tangent vector bundle, whose proof we postpone to the next section.

5.1.1 Statements

The main result of this chapter is the following.

Theorem 5.1.1. *Fix $a \in \mathbb{N}$. There exists $\delta_{N,n,a} \in \mathbb{N}$ such that, if $X \subset \mathbb{P}^N$ is a generic complete intersection of dimension n, codimension c and multidegree $(d_1, \dots, d_c)$ satisfying $c \geqslant n$ and $d_i \geqslant \delta_{N,n,a}$ for all $1 \leqslant i \leqslant c$, then $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)$ is big and there exists a subset $Y \subset X$ of codimension at least two such that*

$$\pi_{\Omega_X}(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a))) \subseteq Y.$$

Remark 5.1.2. It turns out that one can take $\delta_{N,n,a} = \Gamma_{N,n,a+N}$ with the notation of Theorem 4.2.1, and we recall that we gave an effective bound for this number in section 4.2

From this we deduce some other noteworthy conclusions. First, almost everywhere ampleness in the sense of Miyaoka [Miy83] (see also section 1.2.4).

Corollary 5.1.3. *Fix $a \in \mathbb{N}$. If $X \subseteq \mathbb{P}^N$ is a generic complete intersection variety of dimension n, codimension c and multidegree $(d_1, \dots, d_c)$ satisfying $c \geqslant n$ and $d_i \geqslant \delta_{N,n,a+1}$, then $\Omega_X \otimes \mathcal{O}_X(-a)$ is ample modulo an algebraic subset of codimension at least two in X.*

Proof. Applying the theorem we find that $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1)$ is big and that there exists an algebraic subset $Y \subset X$ of codimension 2 in X such that $\pi_{\Omega_X}(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1))) \subseteq Y$. Now take any $\epsilon \leqslant 1$. Take an irreducible curve $C \subset \mathbb{P}(\Omega_X(-a)) = \mathbb{P}(\Omega_X)$ such that

$$\int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X(-a))}(1)) = \int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)) < \epsilon \int_C c_1(\pi^*_{\Omega_X}\mathcal{O}_X(1)) \leqslant \int_C c_1(\pi^*_{\Omega_X}\mathcal{O}_X(1)).$$

This implies that $\int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1)) < 0$. Thus in particular we get $C \subseteq \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1))$ and thus $\pi_{\Omega_X}(C) \subseteq Y$. ☐

In view of Proposition 1.2.15, we also have the announced positive answer to Debarre's conjecture for surfaces (the bound given here is the one we found in section 4.2 page 47).

Corollary 5.1.4. *If $N \geqslant 4$ and $S \subset \mathbb{P}^N$ is a generic complete intersection surface of multidegree $(d_1, \ldots, d_{N-2})$ satisfying $d_i \geqslant \frac{8N+8}{N-3}$, then Ω_S is ample.*

In particular, in $\mathbb{P}^4$, the intersection of two generic hypersurfaces of degree greater than 40 has ample cotangent bundle. To our knowledge this is the first example of surfaces with ample cotangent bundle in $\mathbb{P}^4$, a question that was already raised by Schneider in [Sch92].

5.1.2 Proof

First let us introduce some notation: let $\mathbf{P} := \mathbb{P}^{N_{d_1}} \times \cdots \times \mathbb{P}^{N_{d_c}}$ where $\mathbb{P}^{N_{d_i}} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i))^*)$ be the parameter space, and $\mathcal{X} := \{(x, (t_1, \ldots, t_c)) \in \mathbb{P}^N \times \mathbf{P} \ \ / \ \ t_i(x) = 0 \ \ \forall i\}$ be the universal complete intersection. We will also denote by $\rho_1 : \mathcal{X} \to \mathbb{P}^N$ the projection onto the first factor and $\rho_2 : \mathcal{X} \to \mathbf{P}$. We will use the standard notation $\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_c)$ to denote line bundles on $\mathbf{P}$. Also, write $\pi : \mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \to \mathcal{X}$ for the standard projection. As in [DMR10] the proof of Theorem 5.1.1 is based on Theorem 4.1.1 and on a global generation property that we will prove in section 5.2.

Theorem 5.2.1. *The bundle*

$$T\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}}) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^*\rho_2^*\mathcal{O}_{\boldsymbol{P}}(1, \ldots, 1)$$

is globally generated on $\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}})$.

Observe also the following simple remark.

Remark 5.1.5. Let $X \subset \mathbb{P}^N$ be any projective variety. If $q \geq 0$ then

$$\mathrm{Bs}\left(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi_{\Omega_X}^*\mathcal{O}_X(-a)\right) \subseteq \mathrm{Bs}\left(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi_{\Omega_X}^*\mathcal{O}_X(-a-q)\right).$$

We are now in position to give the proof of Theorem 5.1.1. We will prove that $\delta_{N,n,a} := \Gamma_{N,n,a+N}$ (with the notation of Theorem 4.1.1) will suffice. Fix a multidegree $(d_1, \cdots, d_c)$ such that $d_i \geqslant \delta_{N,n,a} = \Gamma_{N,n,a+N}$ for all $0 \leqslant i \leqslant c$. We start by applying Theorem 4.1.1 to find some $k \in \mathbb{N}$ such that $H^0(X, S^k\Omega_X \otimes \mathcal{O}_X(-ka-kN)) \neq 0$. Using the semicontinuity theorem we find a nonempty open subset $U \subset \mathbf{P}$ such that the restriction map

$$H^0\left(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)\right) \to H^0\left(X_t, S^k\Omega_{X_t} \otimes \mathcal{O}_{X_t}(-ka-kN)\right)$$

is surjective for all $t \in U$, where $\mathcal{X}_U := \rho_2^{-1}(U)$ and $X_t := \rho_2^{-1}(t)$. Fix $t_0 \in U$ and a non-zero section

$$\sigma_0 \in H^0\left(X_{t_0}, S^k\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-ka-kN)\right)$$

and extend it to a section

$$\sigma \in H^0\left(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)\right).$$

Let $\mathcal{Y} := (\sigma = 0) \subset \mathcal{X}_U$. We will prove that

$$\pi_{t_0}\left(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi_{\Omega_{X_{t_0}}}^*\mathcal{O}_{X_{t_0}}(-a-q))\right) \subset Y_{t_0},$$

where $Y_t := X_t \cap \mathcal{Y}$. Denote by $\tilde{\sigma}$ the section in $H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}(-ka-kN)\right)$ corresponding to σ under the canonical isomorphism

$$H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)\right) \simeq H^0\left(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)\right).$$

Let $x \in \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi_{\Omega_{X_{t_0}}}^*\mathcal{O}_{X_{t_0}}(-a))$ so that in particular, thanks to Remark 5.1.5, $x \in (\tilde{\sigma}_{t_0} = 0)$. We will now show that $\pi_{t_0}(x) \in Y_{t_0}$. Take coordinates around x of the form $(t, z_i, [z'_i])$ such that $(t_0, 0, [1 : 0 : \cdots : 0]) = x$. In those coordinates, we write

$$\sigma = \sum_{i_1+\cdots+i_n=k} q_{i_1,\ldots,i_n}(t,z) {z'_1}^{i_1} \cdots {z'_n}^{i_n}.$$

Therefore,

$$(\sigma = 0) = \{(t,z) \ \ / \ \ \forall (i_1,\dots,i_n) \ \ q_{i_1,\dots,i_n}(t,z) = 0\}.$$

Fix any $(i_1,\dots,i_n) \in \mathbb{N}^n$ such that $i_1 + \cdots + i_n = k$. We have to show that $q_{i_1,\dots,i_n}(t_0,0) = 0$. To do so, we apply Theorem 5.2.1 to construct, for each $1 \leqslant j \leqslant n$,

$$V_j \in H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}), T\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \otimes \mathcal{O}_{\mathbb{P}^N}(N) \otimes \mathcal{O}_{\mathbf{P}}(1,\dots,1)\right)$$

such that in our coordinates, $V_j(t_0, 0, [1:0:\cdots:0]) = \frac{\partial}{\partial z'_j}$. By differentiating and contracting by V_j i_j times for each $1 \leqslant j \leqslant n$ we get a new section

$$L_{V_1}\cdots L_{V_1} L_{V_2}\cdots L_{V_n}\tilde{\sigma} \in H^0\left(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka)\right).$$

A local computation gives :

$$L_{V_1}\cdots L_{V_1} L_{V_2}\cdots L_{V_n}\tilde{\sigma}(t_0, 0, [1:0:\cdots:0]) = i_1!\cdots i_n! q_{i_1,\dots,i_n}(t_0,0).$$

But since by hypothesis $x \in \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a))$, we know that

$$L_{V_1}\cdots L_{V_1} L_{V_2}\cdots L_{V_n}\tilde{\sigma}(t_0, 0, [1:0:\cdots:0]) = 0.$$

Therefore we see that $q_{i_1,\dots,i_n}(t_0,0) = 0$, proving our claim.
To complete the proof we just have to show, as in [DT09], the codimension-two refinement. Suppose that Y_{t_0} has a divisorial component E. Since E is effective and $\mathrm{Pic}(X) = \mathbb{Z}$ we can deduce that E is ample. Therefore there is an $m \in \mathbb{N}$ such that mE is very ample. Now take $\sigma^m_{t_0} \in H^0(X, S^{km}\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-mka - mkN))$. The divisorial component of the zero locus of $\sigma^m_{t_0}$ is mE. Now, for any $D \in |mE|$ we get a new section $\sigma^m_{t_0} \otimes D \otimes mE^{-1} \in H^0(X, S^{km}\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-mka - mkN))$. By applying the same argument as above we know that the image of the base locus $\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a))$ lies in the zero locus of this new section $\sigma^m_{t_0} \otimes D \otimes mE^{-1}$ whose divisorial component is D. Thus, since $|mE|$ is base-point free, we know that the image must lie in the codimension at least two part of Y_{t_0}. This concludes the proof.

5.2 Vector fields

As announced during the proof of Theorem 5.1.1, we are now going to prove the global generation property we used.

Theorem 5.2.1. *The bundle*

$$T\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}}) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^*\rho_2^*\mathcal{O}_{\boldsymbol{P}}(1,\dots,1)$$

is globally generated on $\mathbb{P}(\Omega_{\mathcal{X}/\boldsymbol{P}})$.

The proof of this statement is almost the same as the proof of the main Theorem of [Mer09], so there is nothing here that wasn't already in Merker's paper. However since our situation is slightly different we still show how one can adapt Merker's computations here, and in particular we will point out the small differences, and where we are able to gain the better bound on the orders of the poles on has to allow to get global generation. This improvement in the bound is due to the fact that in the situation of [Mer09] the constructed vector fields have to satisfy many equations (as many as the dimension plus one) to be tangent to the higher order jet space, whereas in our situation we only need to go up to jets of order one, thus the constructed vector fields just have to satisfy two equations. Also, for the reader's convenience we adopt the notation of [Mer09].

5.2.1 Notation and coordinates

Now we fix homogeneous coordinates on $\mathbb{P}^N$ and $\mathbb{P}^{N_{d_i}}$ for any $1 \leqslant i \leqslant c$.

$$\begin{array}{rcl} [Z] & = & [Z_0 : \cdots : Z_N] \in \mathbb{P}^N \\ \left[A^i\right] & = & \left[(A^i_\alpha)_{\alpha \in \mathbb{N}^{N+1}, |\alpha| = d_i}\right] \in \mathbb{P}^{N_{d_i}} \end{array}$$

In those coordinates $\mathcal{X} = (F_1 = 0) \cap \cdots \cap (F_c = 0)$ where

$$F_i = \sum_{\substack{\alpha \in \mathbb{N}^{N+1} \\ |\alpha| = d_i}} A^i_\alpha Z^\alpha.$$

To construct vector fields explicitly it will be convenient to work with inhomogeneous coordinates. So from now on we suppose $Z_0 \neq 0$ and $A^i_{(0,d_i,0,\ldots,0)} \neq 0$ and we introduce the corresponding coordinates on $\mathbb{C}^N$ and on $\mathbb{C}^{N_{d_i}}$ by setting $z_i := \frac{Z_i}{Z_0}$ and $a^i_{(\alpha_1,\ldots,\alpha_N)} = \frac{A^i_{(\alpha_0,\ldots,\alpha_N)}}{A^i_{(0,d_i,0,\ldots,0)}}$, where $\alpha_0 = d_i - \alpha_1 - \cdots - \alpha_N$. Now in those coordinates the restriction $\mathcal{X}_0$ of $\mathcal{X}$ to the open subset $\mathbb{C}^N \times \mathbf{P}^0 \subset \mathbb{P}^N \times \mathbf{P}$ where $\mathbf{P}^0 = \mathbb{C}^{N_{d_1}} \times \cdots \times \mathbb{C}^{N_{d_c}}$ is defined by

$$\mathcal{X}_0 = (f_1 = 0) \cap \cdots \cap (f_c = 0),$$

where

$$f_i = \sum_{\substack{\alpha \in \mathbb{N}^{N} \\ |\alpha| \leqslant d_i}} a^i_\alpha z^\alpha.$$

On $\mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ we will use the coordinates $(z_i, a^1_\alpha, \ldots, a^c_\alpha, z'_k)$. Now the equations defining the relative tangent bundle $T_{\mathcal{X}/\mathbf{P}^0} \subset \mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ are

$$f_i = \sum_{\substack{\alpha \in \mathbb{N}^{N} \\ |\alpha| \leqslant d_i}} a^i_\alpha z^\alpha = 0$$

and

$$f'_i = \sum_{\substack{\alpha \in \mathbb{N}^{N} \\ |\alpha| \leqslant d_i}} \sum_{k=1}^{N} a^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z'_k = 0.$$

5.2.2 Vector fields on $T_{\mathcal{X}^0/\mathbf{P}^0}$

Let $\Sigma = \{(z_i, a^1_\alpha, \ldots, a^c_\alpha, z'_k) \ / \ (z'_1, \ldots, z'_N) \neq 0\}$. Following [Mer09] we are going to construct explicit vector fields on $T_{\mathcal{X}^0/\mathbf{P}^0}$ (outside Σ) with prescribed pole order when we look at them as meromorphic vector fields on $T_{\mathcal{X}/\mathbb{P}}$. It will also be clear that the constructed vector fields can actually be viewed as vector fields on $\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})$.

A global vector field on $\mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ is of the form

$$T = \sum_{j=1}^{N} Z_j \frac{\partial}{\partial z_j} + \sum_{\substack{\alpha \in \mathbb{N}^{N} \\ |\alpha| \leqslant d_1}} A^1_\alpha \frac{\partial}{\partial a^1_\alpha} + \cdots + \sum_{\substack{\alpha \in \mathbb{N}^{N} \\ |\alpha| \leqslant d_c}} A^c_\alpha \frac{\partial}{\partial a^c_\alpha} + \sum_{k=1}^{N} Z'_k \frac{\partial}{\partial z'_k}.$$

Such a vector field T is tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ if for all $1 \leqslant i \leqslant c$

$$\left\{ \begin{array}{rcl} T(f_i) & = & 0 \\ T(f'_i) & = & 0 \end{array} \right.$$

First we construct for each $0 \leqslant i \leqslant c$ vector fields of the form

$$\sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant N}} A^i_\alpha \frac{\partial}{\partial a^i_\alpha}.$$

Such a vector field is tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ if

$$T(f_i) = \sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant N}} A^i_\alpha z^\alpha = 0$$

and

$$T(f_i') = \sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant N}} \sum_{k=1}^{N} A^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z_k' = 0.$$

As we are working outside Σ we may as well suppose $z_1' \neq 0$. Set

$$\mathcal{R}_0(z,A) = \sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant N \\ \alpha \neq (0,\ldots,0) \\ \alpha \neq (1,0,\ldots,0)}} A^i_\alpha z^\alpha \quad \textit{and} \quad \mathcal{R}_1(z,A) = \sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant N \\ \alpha \neq (0,\ldots,0) \\ \alpha \neq (1,0,\ldots,0)}} \sum_{k=1}^{N} A^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z_k'.$$

With those notation the tangency property is equivalent to solve the system

$$\begin{cases} A^i_{(0,\cdots,0)} & + & A^i_{(1,0,\ldots,0)} z_1 & + & \mathcal{R}_0(z,A) & = & 0 \\ & & A^i_{(1,0,\ldots,0)} z_1' & + & \mathcal{R}_1(z,A) & = & 0 \end{cases}$$

and as $z_1' \neq 0$ one can solve this in the straightforward way,

$$\begin{cases} A^i_{(1,0,\ldots,0)} & = & \frac{-1}{z_1'} \mathcal{R}_1(z,A) & & \\ A^i_{(0,\ldots,0)} & = & \frac{-z_1}{z_1'} \mathcal{R}_1(z,A) & - & \mathcal{R}_0(z,A) \end{cases}$$

We see that the pole order of vector fields obtained this way is less than N in the z_i. This is where we get the improvement on the pole order. Now in order to span all the other directions, we can take the vector fields constructed by Merker, and the pole order of those fields will be less than N. For the reader's convenience we recall them here, without proof, and refer to [Mer09] for the details.
First we recall how to construct vector fields of higher length in the $\frac{\partial}{\partial a^i_\alpha}$. For any $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leqslant d_i$ and any $\ell \in \mathbb{N}^N$ such that $|\ell| \leqslant N$, set

$$T^\ell_\alpha = \sum_{\substack{\ell' + \ell'' = \ell \\ \ell', \ell'' \in \mathbb{N}^N}} \frac{\ell!}{\ell'! \ell''!} z^{\ell''} \frac{\partial}{\partial a_{\alpha - \ell}}.$$

Those vector fields are of order N in is z_i's, they are also tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ and with the vector fields constructed before, they will span all the $\frac{\partial}{\partial a^i_\alpha}$ directions.

To span the $\frac{\partial}{\partial z_j}$ directions, for all $1 \leqslant j \leqslant N$ we set

$$T_j = \frac{\partial}{\partial z_j} - \sum_{|\alpha| \leqslant d_1 - 1} a^1_{\alpha + e_j} (\alpha_j + 1) \frac{\partial}{\partial a^1_\alpha} \cdots - \sum_{|\alpha| \leqslant d_c - 1} a^c_{\alpha + e_j} (\alpha_j + 1) \frac{\partial}{\partial a^c_\alpha},$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $N-$uple where the only non zero term is in slot j. It is now straightforward to check that those vector fields are tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$. Moreover, they are of order 1 in each of

the a^i_α.
We recall now how to span the $\frac{\partial}{\partial z'_k}$ direction. For any $(\Lambda^\ell_k)^{1\leqslant \ell\leqslant N}_{1\leqslant k\leqslant N} \in GL_N(\mathbb{C})$ we look for vector fields of the form

$$T_\Lambda = \sum_{k=1}^N \left(\sum_{\ell=1}^N \Lambda^\ell_k z'_\ell\right)\frac{\partial}{\partial z'_k} + \sum_{|\alpha|\leqslant d_1} A^1_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^1_\alpha} + \cdots + \sum_{|\alpha|\leqslant d_c} A^c_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^c_\alpha}.$$

Now for each such vector field and for any $1 \leqslant i \leqslant c$ set

$$T^i_\Lambda = \sum_{k=1}^N \left(\sum_{\ell=1}^N \Lambda^\ell_k z'_\ell\right)\frac{\partial}{\partial z'_k} + \sum_{|\alpha|\leqslant d_i} A^i_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^i_\alpha}.$$

Now we can easily check that we have

$$\left\{ \begin{array}{ccc} T_\Lambda(f_i) & = & T^i_\Lambda(f_i) \\ T_\Lambda(f'_i) & = & T^i_\Lambda(f'_i). \end{array} \right.$$

Therefore, to construct vector fields tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$, it is enough to solve those equations for each $1 \leqslant i \leqslant c$ independently, but this can be done using Merker's result on this type of vector fields. By doing so we find for each $1 \leqslant i \leqslant c$ solutions of the type

$$A^i_\alpha(z,a,\Lambda) = \sum_{|\beta|<N} \mathcal{L}^\beta_{\alpha,i}(a,\Lambda) z^\beta,$$

where $\mathcal{L}^\beta_{\alpha,i}$ is bilinear in (a,Λ). Therefore the the constructed fields will have order less than N in the z_j's and of order 1 in each of the a^i_α. We refer again to [Mer09] for a proof of those facts. This leads to the desired result.

Chapter 6

Hyperbolicity for generic complete intersection varieties

In this chapter we use a moving trick that allows us to deduce new hyperbolicity type results from the theorem of Diverio-Merker-Rousseau. In particular we prove that a generic complete intersection variety of sufficiently high multidegree and of big enough codimension is hyperbolic. Our starting point is the results from [DMR10] and [DT09].

We begin with a definition.

Definition 6.0.2. Let X be a projective variety. We define the *algebraic degeneracy locus* to be the Zariski closure of the union of all non-constant entire curves $f : \mathbb{C} \to X$:

$$\mathrm{dl}(X) := \overline{\bigcup f(\mathbb{C})}.$$

Recall the main result proven in [DMR10] and [DT09].

Theorem 6.0.3 (Diverio-Merker-Rousseau, Diverio-Trapani)**.** *For any integer $N \geqslant 2$ there exists $\delta_N \in \mathbb{N}$ such that if $H \subset \mathbb{P}^N$ is a generic hypersurface of degree $d \geqslant \delta_N$, then there exists a proper algebraic subset $Y \subset H$ of codimension at least two in H such that $\mathrm{dl}(H) \subset Y$.*
More precisely, consider the universal hypersurface of degree d in $\mathbb{P}^N$

$$\mathcal{H}_d = \left\{(x,t) \in \mathbb{P}^N \times \mathbb{P}^{N_d} \quad / \quad x \in H_{d,t}\right\},$$

where $\mathbb{P}^{N_d} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^*)$ and $H_{d,t} = (t = 0)$. Denote π_d the projection on the second factor. Then, for $d \geqslant \delta_N$, there exists an open subset $U_d \subset \mathbb{P}^{N_d}$ and an algebraic subset $\mathcal{Y}_d \subset \mathcal{H}_{d|U_d} \subset U_d \times \mathbb{P}^N$ such that for all $t \in U_d$, the fibre $Y_{d,t}$ has codimension 2 in $H_{d,t}$ and $\mathrm{dl}(H_{d,t}) \subset Y_{d,t}$.

Remark 6.0.4. A major result of [DMR10] is that δ_N is effective. They prove that $\delta_N \leqslant 2^{(N-1)^5}$.

6.1 Moving lemma

We are going to use the standard action of $G := \mathrm{Gl}_{N+1}(\mathbb{C})$ on $\mathbb{P}^N$. For any $g \in G$ and any variety $X \subseteq \mathbb{P}^N$ we write $g \cdot X := g^{-1}(X)$.

It is now elementary to see how the degeneracy locus behaves under this action.

Proposition 6.1.1. *Let $g \in G$ and $X \subset \mathbb{P}^N$ a smooth variety. Then $g \cdot \mathrm{dl}(X) = \mathrm{dl}(g \cdot X)$.*

Proof. If $f : \mathbb{C} \to g \cdot X$ is a non-constant entire curve then $g \circ f : \mathbb{C} \to X$ is a non-constant entire curve, therefore $g \circ f(\mathbb{C}) \subseteq \mathrm{dl}(X)$ and thus $f(\mathbb{C}) \subseteq g \cdot \mathrm{dl}(X)$. This proves $\mathrm{dl}(g \cdot X) \subseteq g \cdot \mathrm{dl}(X)$. Similarly we prove the other inclusion. □

Another fact that can be seen immediately is the following

Proposition 6.1.2. *If X_1 and X_2 are two projective varieties in $\mathbb{P}^N$, then* $\mathrm{dl}(X_1 \cap X_2) \subseteq \mathrm{dl}(X_1) \cap \mathrm{dl}(X_2)$.

We can now state our moving lemma.

Lemma 6.1.3. *Let $V \subset \mathbb{P}^N$ and $W \subset \mathbb{P}^N$ be algebraic subsets such that* $\dim(V) = n$ *and* $\dim(W) = m$. *Take $g \in G$ generic.*

1. *If $n + m \geqslant N$ then* $\dim((g \cdot V) \cap W) = n + m - N$.
2. *If $n + m < N$ then $(g \cdot V) \cap W = \emptyset$*

Proof. The proof is done by a dimension count on suitable incidence varieties. First fix any $w \in \mathbb{P}^N$. Consider $G \times V$ with projections q_1 (resp. q_2) on G (resp. V). And consider the incidence variety

$$I_w := \{(g, v) \in G \times V \ / \ g^{-1}(v) = w\} \subseteq G \times V.$$

For any $v \in V$ the fiber $q_{2|I_w}^{-1}(\{v\}) \cong \{g \in G \ / \ g^{-1}(v) = w\}$ is easily seen to be of dimension $(N+1)N + 1$. Therefore $\dim(I_w) = (N+1)^2 - N + n$. Moreover $q_{1|I_w} : I_w \to G$ is injective thus $q_1(I_w) = \{g \in G \ / \ w \in g \cdot V\}$ is of dimension $(N+1)^2 - N + n$.
Now consider $G \times W$ with projections p_1 (resp. p_2) on G (resp. W). Consider the incidence variety

$$I := \{(g, w) \in G \times W \ / \ w \in g \cdot V\} \subseteq G \times W.$$

For any $w \in W$ the fiber $p_{2|I}^{-1}(\{w\}) \cong \{g \in G \ / \ w \in g \cdot V\} = q_1(I_w)$ is of dimension $(N + 1) - N + n$. Therefore $\dim(I) = (N+1)^2 - N + n + m$. Now since $p_{1|I}^{-1}(\{g\}) = g \cdot V \cap W$ and $\dim G = (N+1)^2$ the result follows. In the case $n + m \geqslant N$ observe that for any $g \in G$, $g \cdot V \cap W \neq \emptyset$ and therefore $p_{1|I}$ is surjective. ☐

6.2 Consequences for complete intersection varieties

The strategy is now to use the moving lemma 6.1.3 combined with Propositions 6.1.1 and 6.1.2 to increase the codimension of the degeneracy locus.

This is the content of the following lemma.

Lemma 6.2.1. *Let X be a smooth projective variety. Suppose that there is an algebraic subset $Z \subseteq X$ of codimension $c \geqslant 0$ such that $\mathrm{dl}(X) \subseteq Z$. Fix an ample line bundle A on X. For $e \in \mathbb{N}$ big enough, take $H \in |eA|$ a generic hypersurface. Then there is an algebraic subset $Z' \subset H$ of codimension at least $2 + c$ in H such that $\mathrm{dl}(H) \subseteq Z'$.*

Proof. As we are looking at the situation for e big enough we might as well suppose that A is very ample. We use then A to embed $X \subseteq \mathbb{P}^N$. Under this embedding we have the identification $A \cong \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$. Take $d \geqslant \delta_N$ so that we can apply Theorem 6.0.3 in $\mathbb{P}^N$. Moreover, take d big enough to have

$$H^1(\mathbb{P}^N, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0.$$

Therefore one has a surjection

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(X, \mathcal{O}_X(d)) \to 0.$$

We are therefore able to extend all the hypersurfaces we are interested in to hypersurfaces of $\mathbb{P}^N$.

We decompose the rest of the proof into three assertions.

Assertion 1. A generic hypersurface $D \in |\mathcal{O}_X(d)|$ can be extended to a generic hypersurface $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$. More precisely, for any non-empty open subset $U \subseteq |\mathcal{O}_{\mathbb{P}^N}(d)|$, there exists an non-empty open subset

$U_X \subseteq |\mathcal{O}_X(d)|$, such that for any $D \in U_X$ there exists $H \in U$ such that $D = X \cap H$.

Assertion 2. For a generic hypersurface $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$ and a generic $g_H \in Gl_{N+1}(\mathbb{C})$ (depending on H) there exists an algebraic subset $Z' \subset X \cap g_H \cdot H$ of codimension at least $2+c$ such that $\mathrm{dl}(X \cap g_H \cdot H) \subseteq Z'$.

Assertion 3. For a generic $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$ there exists an algebraic subset $Z' \subset X \cap H$ of codimension at least $2+c$ such that $\mathrm{dl}(X \cap H) \subseteq Z'$.

The lemma then clearly follows from assertion 1 and 3. The subtlety of this lemma is the precise meaning of "generic" at each step.

Proof of assertion 1. Let $U \subseteq \mathbb{P}^{N_d} = |\mathcal{O}_{\mathbb{P}^N}(d)|$ be a non-empty open subset which contains the genericity assumption (H is generic if $H \in U$). We have to prove that there is an open subset

$$U_X \subseteq \mathbb{P}^{N_d(X)} := |\mathcal{O}_X(d)|$$

such that any $D \in U_X$ can be extended to an element $H \in U$. Let $W := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(d))^*)$. We therefore obtain a surjective map,

$$\mathbb{P}^{N_d} \setminus W \xrightarrow{\pi} \mathbb{P}^{N_d(X)}.$$

Let $F := \mathbb{P}^{N_d} \setminus U$, and $F' := F \setminus W$. And consider the restriction $\pi_{F'} : F' \to \mathbb{P}^{N_d(X)}$. Consider also the blow-up $\mathrm{Bl}_W \mathbb{P}^{N_d}$. It comes with a proper map $\tilde{\pi} : \mathrm{Bl}_W \mathbb{P}^{N_d} \to \mathbb{P}^{N_d(X)}$ whose fibers are irreducible of dimension $N_d - N_d(X)$. Let $\widetilde{F} \subset \mathrm{Bl}_W \mathbb{P}^{N_d}$ be the strict transform of F in the blow-up. Now we look at the proper map $\tilde{\pi}_F : \widetilde{F} \to \mathbb{P}^{N_d(X)}$, by the semi-continuity theorem we see that the application $x \mapsto \dim \tilde{\pi}_F^{-1}(\{y\})$ is upper semi continuous, therefore the set $F_X := \{y \in \mathbb{P}^{N_d(X)} \ / \ \tilde{\pi}^{-1}(\{y\}) \subseteq \widetilde{F}\} = \{y \in \mathbb{P}^{N_d(X)} \ / \ \dim \tilde{\pi}_F^{-1}(\{y\}) = \dim \mathbb{P}^{N_d} - \dim \mathbb{P}^{N_d(X)}\}$ is a closed subset of $\mathbb{P}^{N_d(X)}$. On the other hand, we observe that $F_X = \{y \in \mathbb{P}^{N_d(X)} \ / \ \dim \pi_{F'}^{-1}(\{y\}) = \dim \mathbb{P}^{N_d} - \dim \mathbb{P}^{N_d(X)}\}$. Moreover $F_X \neq \mathbb{P}^{N_d(X)}$ because $F \neq \mathbb{P}^{N_d}$. Therefore we can define $U_X := \mathbb{P}^{N_d(X)} \setminus F_X$ which is then an nonempty open subset that satisfies the expected property.

Proof of assertion 2. Take a generic $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$. By Theorem 6.0.3 we know that there exists $Y \subset H$ such that $\mathrm{dl}(H) \subset Y$. Then applying lemma 6.1.3 to Z and Y combined with remarks 6.1.2 and 6.1.1 yields the expected result.

Proof of assertion 3. We take the notation of Theorem 6.0.3. Consider the family $\mathcal{Y}'_d := \mathcal{Y}_d \cap pr_1^{-1} Z \subset \mathcal{H}_d \cap pr_1^{-1} X \subset U \times X$. The application $t \mapsto \dim(Y'_{d,t})$ is upper semi-continuous. Therefore, $W := \{t \in U / \dim(Y'_{d,t}) > n - c - 2\}$ is a closed subset. Applying assertion 2 tells us that the complement is non-empty and therefore, an open dense subset.

□

From this lemma, Theorem E follows as a straightforward induction.

Theorem E. *Let M be a smooth N-dimensional projective variety. Take $A_1, \ldots, A_c$ ample line bundles on M. For $d_1, \ldots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1 A_1|, \ldots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then, there exists an algebraic subset $Z \subset X$ of codimension at least $2c$ such that $\mathrm{dl}(X) \subseteq Z$. In particular if $2c \geqslant n$ then X is hyperbolic.*

Remark 6.2.2. The case $M = \mathbb{P}^N$ is of particular interest to us. In this situation we let $A_1 = \cdots = A_c = \mathcal{O}_{\mathbb{P}^N}(1)$ which is very ample. Doing the proof in this particular setting we see that it is sufficient to take $d_i \geqslant \delta_N$ to have the conclusions of Theorem A, where δ_N is any bound that holds in Diverio-Merker-Rousseau's theorem, for example $2^{(N-1)^5}$.

Remark 6.2.3. To obtain hyperbolicity when $2\,\mathrm{codim}_M(X) \geqslant \dim(X)$, we used the codimension 2 refinement of Diverio and Trapani [DT09]. However, using just the initial result of [DMR10] we could have used the exact same trick to obtain hyperbolicity as soon as $\mathrm{codim}_M(X) \geqslant \dim(X)$.

Chapter 7

Symmetric differential form computations

One of the main difficulties of Debarre's conjecture in projective spaces compared with Debarre's theorem in abelian varieties is somehow the lack of a suitable, and usable, geometric setting. Let us be more precise. Let A be an N-dimensional abelian variety and $X \subseteq A$ a subvariety. Then we have a composite map,

$$\mathbb{P}(\Omega_X) \hookrightarrow \mathbb{P}(\Omega_A) = \mathbb{P}^N \times A \to \mathbb{P}^N.$$

This map just comes from the fact that the tangent bundle to an abelian variety is trivial. Then it is easy to convince ourself that Ω_X is ample if and only if the map $\phi : \mathbb{P}(\Omega_X) \to \mathbb{P}^N$ is finite. This is the starting point of Debarre's proof of his theorem in abelian varieties. However there is no such map in projective spaces, and therefore it is not possible to adapt his proof in this new situation. But one can still wonder what could be a good geometric setting from which we could gain some information. It turns out there are two somehow natural way to do this. First it can be noticed that if $X \subset \mathbb{P}^N$ is a smooth subvariety then $\mathbb{P}(\Omega_X(2))$ can be interpreted as some incidence variety. We left the twist by two to emphasize that we don't get an information on Ω_X but on $\Omega_X(2)$. We then get a map to some grassmannian that we can use. The second interesting geometric setting is based on the Gauss map. If $X \subseteq \mathbb{P}^N$ is an n-dimensional subvariety, we consider

$$\begin{array}{ccc} X & \stackrel{\gamma_X}{\to} & \mathrm{Gr}(n, \mathbb{P}^N) \\ x & \mapsto & \mathbb{T}_x X \end{array}$$

and study the positivity of the pull-back of the tautological bundle on $\mathrm{Gr}(n, \mathbb{P}^N)$ under γ_X. This was already studied by Debarre.

In this chapter we will look at what we can deduce from these two approaches. We start by giving a geometric interpretation of the ampleness of $\Omega_X(2)$. It turns out that this controls the existence of lines on X.
Then we study more closely the Gauss map. Another interpretation of this was introduced by Bogomolov and De Oliveira in [BDO08]. They introduce and study the bundle $\widetilde{\Omega}_X$ of differential forms on $\widehat{X}$, the cone over X, which are invariant under the $\mathbb{C}^*$-action but which do not necessarily satisfy the Euler condition (this turns out to be just a twist of the pull-back of the tautological bundle under the Gauss map). Because this bundle is easier to deal with (we do not have to care about the Euler condition), we will use it here as a computational tool to study the bundle Ω_X. This will reinterpret the problem of understanding symmetric differential forms on a complete intersection variety into a purely combinatorial issue. We will therefore be able to compute some explicit symmetric differential forms on some particular complete intersection varieties. The time being, we do not know to which extent those computations could lead to noteworthy conclusions. However, we strongly believe it is an important issue to have access to a way of producing explicit equations for symmetric differential forms.

7.1 Ampleness of $\Omega_X(2)$

We know that $\Omega_X(2)$ is nef since it is globally generated. Therefore, it seems a natural question to ask for which varieties this bundle is ample. It turns out that this has a simple geometric interpretation, which is that $\Omega_X(2)$ is ample if and only if there are no lines in X, as we shall now see.
Fix an $(N+1)$-dimensional complex vector space V, denote by $\mathbb{P}^N = \mathbb{P}(V^*)$ the projectivized space of lines in V, by $p : V \setminus \{0\} \to \mathbb{P}(V^*)$ the projection, and by $\mathrm{Gr}(2, V) = \mathrm{Gr}(1, \mathbb{P}^N)$ the space of vector planes in V which is also the space of lines in $\mathbb{P}^N$. We will also consider the projection $\pi : \mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathbb{P}^N$. The key point is the following lemma, which was pointed out to us by Frédéric Han.

Lemma 7.1.1. *There is a map $\varphi : \mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathbb{P}(\Lambda^2 V^*)$ such that $\varphi^*\mathcal{O}_{\mathbb{P}(\Lambda^2 V^*)}(1) = \mathcal{O}_{\Omega_{\mathbb{P}^N}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^N}(2)$. Moreover this application factors through the Plücker embedding $Gr(2, \mathbb{P}^N) = Gr(2, V) \hookrightarrow \mathbb{P}(\Lambda^2 V^*)$. More precisely, an element $(x, [\xi]) \in \mathbb{P}(\Omega_{\mathbb{P}^N})$ with $x \in \mathbb{P}^N$ and $\xi \in T_x\mathbb{P}^N$, gets mapped to the unique line Δ in $\mathbb{P}^N$ satisfying $\xi \in T_x\Delta \subseteq T_x\mathbb{P}^N$.*

Proof. Take the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to V \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to T\mathbb{P}^N \to 0$$

and apply Λ^{N-1} to it in order to get the quotient

$$\Lambda^{N-1}V \otimes \mathcal{O}_{\mathbb{P}^N}(N-1) \to \Lambda^{N-1}T\mathbb{P}^N \to 0.$$

Now using the well-known dualities, $\Lambda^{N-1}V = \Lambda^2 V^*$ and $\Lambda^{N-1}T\mathbb{P}^N = \Omega_{\mathbb{P}^N} \otimes K^*_{\mathbb{P}^N} = \Omega_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(N+1)$, and tensoring everything by $\mathcal{O}_{\mathbb{P}^N}(1-N)$, we get

$$\Lambda^2 V^* \to \Omega_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(2) \to 0.$$

This yields the map $\varphi : \mathbb{P}(\Omega_{\mathbb{P}^N}) = \mathbb{P}(\Omega_{\mathbb{P}^N}(2)) \hookrightarrow \mathbb{P}^N \times \mathbb{P}(\Lambda^2 V^*) \to \mathbb{P}(\Lambda^2 V^*)$ such that $\varphi^*\mathcal{O}_{\mathbb{P}(\Lambda^2 V^*)}(1) = \mathcal{O}_{\Omega_{\mathbb{P}^N}(2)}(1) = \mathcal{O}_{\Omega_{\mathbb{P}^N}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^N}(2)$.

To see the geometric interpretation of this map, it suffices to backtrack through the previous maps. Take a point $x \in \mathbb{P}^N$ and a vector $0 \neq \xi \in T_x\mathbb{P}^N$ and fix a basis $(\xi_0, \ldots, \xi_{N-1})$ of $T_x\mathbb{P}^N$ such that $\xi_0 = \xi$. Now take $v \in V$ such that $p(v) = x$ and a basis $(e_0, \ldots, e_N)$ of $T_vV = V$ such that $d_vp(e_N) = 0$ and $d_vp(e_i) = \xi_i$ for $i < N$. We just have to check that $(x, [\xi])$ is mapped to the announced line Δ which, with our notation, corresponds to the point $[e_0 \wedge e_N] \in \mathbb{P}(\Lambda^2 V^*)$. This is easily verified as the above maps can be described explicitly, as follows:

$$\begin{array}{ccccccc} \mathbb{P}(\Omega_{\mathbb{P}^N,x}) & \to & \mathbb{P}(\Lambda^{N-1}T_x\mathbb{P}^N) & \to & \mathbb{P}(\Lambda^{N-1}V) & \to & \mathbb{P}(\Lambda^2 V^*) \\ [\xi_0] & \mapsto & \left[\xi_1^* \wedge \cdots \wedge \xi_{N-1}^*\right] & \mapsto & \left[e_1^* \wedge \cdots \wedge e_{N-1}^*\right] & \mapsto & [e_0 \wedge e_N]. \end{array}$$

□

With this we can prove our proposition,

Proposition 7.1.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety. Then $\Omega_X(2)$ is ample if and only if X does not contain any line.*

Proof. By Lemma 7.1.1, we know that $\Omega_X(2)$ is ample if and only if the restriction $\varphi_X : \mathbb{P}(\Omega_X) \subseteq \mathbb{P}(\Omega_{\mathbb{P}^N}) \to Gr(2, \mathbb{P}^N)$ of φ is finite.

Now, if X contains a line Δ then φ_X is not finite since the curve $\mathbb{P}(K_\Delta) \subseteq \mathbb{P}(\Omega_X)$ gets mapped to the point in $Gr(2, \mathbb{P}^N)$ representing Δ.

If φ_X is not finite then there is a curve $C \subseteq \mathbb{P}(\Omega_X)$ which gets mapped to a point in $Gr(2, \mathbb{P}^N)$ corresponding to a line Δ in $\mathbb{P}^N$. Let $\Gamma = \pi(C)$, Lemma 7.1.1 tells us that the embedded tangent space $\mathbb{T}_x\Gamma$ equals Δ for all $x \in \Gamma$ and therefore $\Delta \subseteq X$. □

Remark that by a dimension count on the incidence variety of lines on the universal hypersurface one can see that a generic hypersurface $H \subset \mathbb{P}^N$ of degree $d > 2N-3$ contains no line, and therefore $\Omega_H(2)$ is ample. In particular we have,

Corollary 7.1.3. *If $X = H_1 \cap \cdots \cap H_C \subset \mathbb{P}^N$ is a generic complete intersection variety such that there exists $1 \leqslant i \leqslant c$ for which $\deg(H_i) > 2N-3$, then $\Omega_X(2)$ is ample.*

7.2 The Gauss map and the $\widetilde{\Omega}_X$ bundle

We start with some preliminaries on the Gauss map and on the $\widetilde{\Omega}_X$ bundle. We refer to [BDO08] for very interesting applications of this bundle. Let V be an $(N+1)$-dimensional vector space with a basis $(e_0, \dots, e_N)$. Let $\mathbb{P}^N := P(V) = \mathbb{P}(V^*)$. Let $X \subseteq \mathbb{P}^N$ be an n-dimensional smooth subvariety. We denote by γ_X the Gauss map

$$\begin{array}{rcl} X & \to & \mathrm{Gr}(n, \mathbb{P}^N) \\ x & \mapsto & \mathbb{T}_x X, \end{array}$$

where $\mathbb{T}_x X$ denotes denotes the embedded tangent space to X at x in $\mathbb{P}^N$. We denote the rank-$(n+1)$ tautological vector bundle on the grassmannian by $\mathcal{S}_{n+1}$. We then define

$$\widetilde{\Omega}_X := \gamma^* \mathcal{S}_{n+1}^* \otimes \mathcal{O}_X(-1).$$

The first observation that one can make is that one has a natural identification

$$\widetilde{\Omega}_{\mathbb{P}^N} \cong \bigoplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}^N}(-1) dX_i.$$

The main properties of that bundle that we will use can be summarized in the following.

Proposition 7.2.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth algebraic subvariety. Let $Y := (F = 0) \cap X$, where $F \in \mathbb{C}[X_0, \dots, X_N]$ is a homogenous polynomial of degree d such that Y is a smooth hypersurface of X. We then have two exact sequences:*

$$0 \to \Omega_X \to \widetilde{\Omega}_X \to \mathcal{O}_X \to 0,$$

which is just the Euler exact sequence, and

$$0 \to \mathcal{O}_Y(-d) \stackrel{\cdot dF}{\to} \widetilde{\Omega}_{X_{|Y}} \to \widetilde{\Omega}_Y \to 0,$$

the tilde conormal exact sequence, where the first map is just multiplication by dF.

Proof. We start by the Euler exact sequence. Let $\widehat{X} \subseteq \mathbb{C}^{N+1}$ be the cone over X and let $\rho_X : \widehat{X} \to X$ be the projection. Now $\rho_X^* \gamma_X^* \mathcal{S}_{n+1} = T\widehat{X}$. If we look at the differential, $d\rho_X : T\widehat{X} \to \rho_X^* TX$ we see that this morphism is not invariant under the $\mathbb{C}^*$-action on $\mathbb{C}^{N+1}$. As, if $x \in \mathbb{C}^{N+1}$, $\xi \in T_x$ and $\lambda \in \mathbb{C}^*$ we have $d\rho_{X,\lambda x} \xi = \frac{1}{\lambda} d\rho_{X,x} \xi$. But we can compensate this behavior by a twist by $\mathcal{O}_X(-1)$ as in the following:

$$\begin{array}{rcl} \gamma_X^* \mathcal{S}_{n+1,x} & \to & T_x X \otimes \mathcal{O}_{X,x}(-1) \\ (x, \xi) & \mapsto & (x, d\rho_{X,x} \xi \otimes x). \end{array}$$

This yields the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \gamma_X^* \mathcal{S}_{n+1} \to TX(-1) \to 0.$$

To conclude, we have just to twist by $\mathcal{O}_X(1)$ and take the dual.

One can observe that on has a commutative diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_Y(-d) \arrow[r, equal] \arrow[d] & \mathcal{O}_Y(-d) \arrow[d] & & \\
0 \arrow[r] & \Omega_{X_{|Y}} \arrow[r] \arrow[d] & \widetilde{\Omega}_{X_{|Y}} \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_Y \arrow[r] \arrow[d] & \widetilde{\Omega}_Y \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The only thing we have to check for the tilde conormal exact sequence is that the application $\mathcal{O}_X(-d) \to \widetilde{\Omega}_{X_{|Y}}$ is just multiplication by dF. We first explain what this means exactly. Start with the map

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^N}(-d) & \stackrel{\cdot dF}{\to} & \widetilde{\Omega}_{\mathbb{P}^N} = \bigoplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}^N} dX_i \\
\xi & \mapsto & \sum_{i=0}^{N} \xi \dfrac{\partial F}{\partial X_i} dX_i,
\end{array}$$

then factor through the quotient by the ideal of X and compose it with the restriction map

$$\mathcal{O}_X(-d) \stackrel{\cdot dF}{\to} \widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to \widetilde{\Omega}_X.$$

Then quotient out by F to obtain a map $\mathcal{O}_Y(-d) \to \widetilde{\Omega}_{X_{|Y}}$. It then suffices to check that for any $y := [y_0 : \cdots : y_N] \in Y$ and for any $\xi \in \mathcal{O}_Y(-d)_y$ the form $\sum_i \xi \frac{\partial F}{\partial X_i} dX_i$ is identically 0 when restricted to $\mathbb{T}_y Y$. But this is immediate using $\mathbb{T}_y Y = \mathbb{T}_y X \cap \{[a_0 : \cdots : a_N] \ / \ \sum_{i=0}^{N} a_i \frac{\partial F}{\partial X_i}(y) = 0\}$. $\square$

Observe that $\widetilde{\Omega}_X(1)$ is nef since it is the quotient of a trivial bundle ($\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(1) = \bigoplus_{i=0}^{N} \mathcal{O}_X dX_i$). But $\widetilde{\Omega}_X$ is never ample since it has a trivial quotient. Also, we mention that Debarre proved that under the hypothesis of his conjecture, $\widetilde{\Omega}_X(1)$ is ample ([Deb05] Theorem 16).

Remark 7.2.2. We would like to emphasize what we used during the proof. Even though the bundle $\widetilde{\Omega}_Y$ does not extend as a vector bundle on $\mathbb{P}^N$, the map $\mathcal{O}_X(-d) \stackrel{\cdot dF}{\to} \widetilde{\Omega}_{X_{|Y}}$ does extend to a map $\mathcal{O}_{\mathbb{P}^N}(-d) \stackrel{\cdot dF}{\to} \widetilde{\Omega}_{\mathbb{P}^N} = \bigoplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}^N} dX_i$ This is a major advantage of the $\widetilde{\Omega}$ bundles compared to the Ω bundles.

7.3 Cohomology vanishing for $S^k\widetilde{\Omega}_X$

We continue by a study of the cohomology groups of the symmetric powers of $\widetilde{\Omega}_X$. The arguments are just long exact sequence arguments but we give the details since it gives some insight in our strategy to construct explicit global symmetric differential forms on complete intersection varieties.

Let us start with a first vanishing theorem.

Theorem 7.3.1. *Let $X = H_1 \cap \dots \cap H_c \subset \mathbb{P}^N$ be a complete intersection variety of dimension n and codimension c. And let $H \subset X$ be a smooth degree d hypersurface. Then,*

1. *$\forall a > 0$ and $\forall i \leqslant N - c - 1$ $H^i(X, \mathcal{O}_X(-a)) = 0$.*
2. *$\forall k \geqslant 1$, $\forall a \geqslant 0$ and $\forall i \leqslant N - 2c - 1 = n - c - 1$ $H^i(X, S^k\widetilde{\Omega}_X(-a)) = 0$.*
3. *$\forall k \geqslant 1$, $\forall a \geqslant 0$ and $\forall i \leqslant N - 2c - 2 = n - c - 2$ $H^i(H, S^k\widetilde{\Omega}_{X_{|H}}(-a)) = 0$.*

Proof. We start by 1). This is just an induction on c.
When $c = 0$ this is a well know fact (see [Har77], chap III theorem 5.11).
Let $c > 0$ and suppose the statement holds for all $c' < c$. Denote $X = X' \cap H_c$ with $X' = H_1 \cap \dots \cap H_{c-1}$. Take the twisted restriction exact sequence

$$0 \to \mathcal{O}_{X'}(-d_c - a) \to \mathcal{O}_{X'}(-a) \to \mathcal{O}_X(-a) \to 0$$

and look at the associated long exact sequence in cohomology

$$\begin{aligned}\cdots &\to H^{N-c-1}(X', \mathcal{O}_{X'}(-d_c - a)) \to H^{N-c-1}(X', \mathcal{O}_{X'}(-a)) \to H^{N-c-1}(X, \mathcal{O}_X(-a)) \to \\ &\to H^{N-c}(X', \mathcal{O}_{X'}(-d_c - a)) \to H^{N-c}(X', \mathcal{O}_{X'}(-a)) \to H^{N-c}(X, \mathcal{O}_X(-a)) \to \\ &\to H^{N-c+1}(X', \mathcal{O}_{X'}(-d_c - a)) \to H^{N-c+1}(X', \mathcal{O}_{X'}(-a)) \to H^{N-c+1}(X, \mathcal{O}_X(-a)) \to \cdots\end{aligned}$$

Using our induction hypothesis on X', we get that $H^i(X', \mathcal{O}_{X'}(-d_c - a)) = H^i(X', \mathcal{O}_{X'}(-a)) = 0$ for all $i \leqslant N - c$ and therefore we get $H^i(X, \mathcal{O}_X(-a)) = 0$ for all $i \leqslant N - c - 1$.

Now we prove 2) and 3). Again by induction on c. Let $c = 0$, $k \geq 1$ and $a \geqslant 0$.

$$\begin{aligned}H^i(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(-a)) &= S^k\mathbb{C}^{N+1} \otimes H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-k - a)) \\ &= 0 \quad \forall \; i < N. \end{aligned} \tag{7.1}$$

For the second statement, let $d := \deg(H)$. Observe that the restriction exact sequence yields

$$0 \to S^k\widetilde{\Omega}_{\mathbb{P}^N} \otimes \mathcal{O}_X(-a - d) \to S^k\widetilde{\Omega}_{\mathbb{P}^N} \otimes \mathcal{O}_X(-a) \to S^k\widetilde{\Omega}_{\mathbb{P}^N_{|H}} \otimes \mathcal{O}_H(-a) \to 0.$$

Using (7.1) we see that the associated long exact sequence is reduced to

$$0 \to H^{N-1}(H, S^k\Omega_{\mathbb{P}^N_{|H}}) \to H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a - d)) \to H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)) \to 0,$$

which proves that $H^i(H, S^k\widetilde{\Omega}_{\mathbb{P}^N_{|H}}(-a)) = 0$ for all $i \leqslant N - 2$.

The rest of the argument is similar. We write it down for the sake of completeness.
Let $c > 0$ and suppose the statement holds for all $c' < c$. As before, let $X = X' \cap H_c$. Now if we look at the twisted symmetric powers of the conormal exact sequence

$$0 \to S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a - d_c) \to S^k\widetilde{\Omega}_{X'_{|X}}(-a) \to S^k\widetilde{\Omega}_X(-a) \to 0,$$

we get the following long exact sequence in cohomology.

$$\begin{aligned}\cdots &\to H^{n-c-1}(S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a - d_c)) \to H^{n-c-1}(S^k\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-c-1}(S^k\widetilde{\Omega}_X(-a)) \to \\ &\to H^{n-c}(S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a - d_c)) \to H^{n-c}(S^k\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-c}(S^k\widetilde{\Omega}_X(-a)) \to \\ &\to H^{n-c+1}(S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a - d_c)) \to H^{n-c+1}(S^k\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-c+1}(S^k\widetilde{\Omega}_X(-a)) \to \cdots\end{aligned}$$

By induction hypothesis we know that $H^i(X, S^k\widetilde{\Omega}_{X'_{|X}}(-a)) = H^i(X, S^k\widetilde{\Omega}_{X'_{|X}}(-a - d_c)) = 0$ for all $i \leqslant N - 2(c-1) - 2 = N - 2c = n - c$ (note that in the case $k = 1$, this also holds by 1)) and therefore,

$H^i(X, S^k\widetilde{\Omega}_X(-a)) = 0$ for all $i \leqslant n-c-1 = N-2c-1$. To see the other statement, look at the restriction exact sequence tensored by $S^k\widetilde{\Omega}_X(-a)$:

$$0 \to S^k\widetilde{\Omega}_X(-a-d) \to S^k\widetilde{\Omega}_X(-a) \to S^k\widetilde{\Omega}_{X_{|H}}(-a) \to 0$$

and take the associated long exact sequence

$$\begin{aligned}\cdots \to & H^{n-c-2}(S^k\widetilde{\Omega}_X(-a-d)) \to H^{n-c-2}(S^k\widetilde{\Omega}_X(-a)) \to H^{n-c-2}(S^k\widetilde{\Omega}_{X_{|H}}(-a)) \to \\ \to & H^{n-c-1}(S^k\widetilde{\Omega}_X(-a-d)) \to H^{n-c-1}(S^k\widetilde{\Omega}_X(-a)) \to H^{n-c-1}(S^k\widetilde{\Omega}_{X_{|H}}(-a)) \to \\ \to & \ H^{n-c}(S^k\widetilde{\Omega}_X(-a-d)) \ \to \ H^{n-c}(S^k\widetilde{\Omega}_X(-a)) \ \to \ H^{n-c}(S^k\widetilde{\Omega}_{X_{|H}}(-a)) \ \to \cdots\end{aligned}$$

We just proved that $H^i(S, S^k\widetilde{\Omega}_X(-a)) = H^i(S, S^k\widetilde{\Omega}_X(-a-d)) = 0$ for all $i \leqslant n-c-1$. Therefore, we get $H^i(H, S^k\widetilde{\Omega}_{X_{|H}}(-a)) = 0$ for all $i \leqslant n-c-2 = N-2c-2$.

□

From this we deduce two corollaries. First, by looking for the first non zero term in the above long exact sequences we get,

Corollary 7.3.2. *With the notation of the theorem, we have isomorphisms*

$$\begin{aligned} H^{N-2c-1} \quad & (H, S^k\widetilde{\Omega}_{X_{|H}}(-a)) \equiv \\ \ker \quad & \Big(H^{N-2c}(X, S^k\widetilde{\Omega}_X(-a-d)) \to H^{N-2c}(X, S^k\widetilde{\Omega}_X(-a))\Big) \\ H^{N-2c-2} \quad & (H, S^k\widetilde{\Omega}_H(-a)) \equiv \\ \ker \quad & \Big(H^{N-2c-1}(X, S^k\widetilde{\Omega}_{X_{|H}}(-a-d)) \to H^{N-2c-1}(X, S^k\widetilde{\Omega}_{X_{|H}}(-a))\Big). \end{aligned}$$

By composing the coboundary maps appearing above we get,

Corollary 7.3.3. *Let $D := d_1 + \cdots + d_c$. Then we have the injections*

$$\begin{aligned} H^{N-2c}(X, S^{k+c}\widetilde{\Omega}_X(-a)) \quad & \hookrightarrow \quad H^{N-2c+1}(X', S^{k-1+c}\widetilde{\Omega}_{X'_{|X}}(-a-d_c)) \\ & \hookrightarrow \quad H^{N-2c+2}(X', S^{k-1+c}\widetilde{\Omega}_{X'}(-a-2d_c)) \\ & \hookrightarrow \\ & \cdots \\ & \hookrightarrow \quad H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D)). \end{aligned}$$

It turns out that, if we are more careful during the long exact sequences computations, we can have a more refined statement.

Theorem 7.3.4. *With the same notation as previously, let $c > 0$ and $a \geqslant 0$. Then,*

$$H^\ell(X, S^k\widetilde{\Omega}_X(-a)) = 0 \ \ \forall k \geqslant 1, \ \ell \geqslant 0 \ \ \textit{satisfying} \ \ k + \ell \leqslant N - c - 1 = n - 1.$$

Proof. The idea is the same as before, but we just look more carefully at what happens. We proceed by induction. If $c = 1$, this is contained in Theorem 7.3.1. Suppose now that $c > 1$ and that the statement holds for all $c' < c$. Let $X = X' \cap H_c$ where $X' = H_1 \cap \cdots \cap H_{c-1}$. Using the induction hypothesis on X', we get

$$H^{\ell'}(X', S^{k'}\widetilde{\Omega}_{X'}(-a)) = 0 \ \ \forall k' \geqslant 1, \ \ell' \geqslant 0 \quad \textit{satisfying} \ \ k' + \ell' \leqslant N - (c-1) - 1 = N - c = n.$$

From this we will first study the cohomology of $S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)$.

Look at the restriction exact sequence tensored by $S^{k'}\widetilde{\Omega}_X(-a)$:

$$0 \to S^{k'}\widetilde{\Omega}_{X'}(-a-d_c) \to S^{k'}\widetilde{\Omega}_X(-a) \to S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a) \to 0.$$

The associated long exact sequence in cohomology is then

$$\begin{aligned}\cdots &\to H^{n-k'-1}(S^{k'}\widetilde{\Omega}_{X'}(-a-d_c)) \to H^{n-k'-1}(S^{k'}\widetilde{\Omega}_{X'}(-a)) \to H^{n-k'-1}(S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)) \to \\ &\to H^{n-k'}(S^{k'}\widetilde{\Omega}_{X'}(-a-d_c)) \to H^{n-k'}(S^{k'}\widetilde{\Omega}_{X'}(-a)) \to H^{n-k'}(S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)) \to \\ &\to H^{n-k'+1}(S^{k'}\widetilde{\Omega}_{X'}(-a-d_c)) \to H^{n-k'+1}(S^{k'}\widetilde{\Omega}_{X'}(-a)) \to H^{n-k'+1}(S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)) \to \cdots\end{aligned}$$

We know that $H^{\ell'}(X', S^{k'}\widetilde{\Omega}_{X'}(-a)) = 0$ for all $\ell' \leqslant n-k'$, thus we get $H^{\ell'}(X', S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)) = 0$ for all $\ell' \leqslant n-k'-1$. This can be summarized by,

$$H^{\ell'}(X', S^{k'}\widetilde{\Omega}_{X'_{|X}}(-a)) = 0 \quad \forall k' \geqslant 1,\ \ell' \geqslant 0 \quad \textit{satisfying} \quad k'+\ell' \leqslant n-1 = N-c-1. \tag{7.2}$$

Now from this we can deduce the full statement. First we look at the case $k=1$. We have to prove that $H^{\ell}(X, \widetilde{\Omega}_X(-a)) = 0$ for all $\ell \leqslant n-2$.

The conormal bundle exact sequence

$$0 \to \mathcal{O}_X(-d_c-a) \to \widetilde{\Omega}_{X'_{|X}}(-a) \to \widetilde{\Omega}_X(-a) \to 0$$

yields the long exact sequence

$$\begin{aligned}\cdots &\to H^{N-c-3}(\mathcal{O}_X(-a-d_c)) \to H^{N-c-3}(\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{N-c-3}(\widetilde{\Omega}_X(-a)) \to \\ &\to H^{N-c-2}(\mathcal{O}_X(-a-d_c)) \to H^{N-c-2}(\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{N-c-2}(\widetilde{\Omega}_X(-a)) \to \\ &\to H^{N-c-1}(\mathcal{O}_X(-a-d_c)) \to H^{N-c-1}(\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{N-c-1}(\widetilde{\Omega}_X(-a)) \to \cdots\end{aligned}$$

Applying Theorem 7.3.1 we get $H^{\ell}(X, \mathcal{O}_X(-a-d_c)) = 0$ for all $\ell \leqslant N-c-1$, and applying (7.2) we get $H^{\ell}(X, \widetilde{\Omega}_{X'_{|X}}(-a)) = 0$ for all $\ell \leqslant N-c-2$. Combining this with the long exact sequence, we get $H^{\ell}(X, \widetilde{\Omega}_X(-a)) = 0$ for all $\ell \leqslant N-c-2 = n-2$, and this is what we wanted.

Now we let $k \geqslant 2$.

We have the exact sequence

$$0 \to S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-d_c-a) \to S^{k}\widetilde{\Omega}_{X'_{|X}}(-a) \to S^{k}\widetilde{\Omega}_X(-a) \to 0.$$

From this we have the long exact sequence

$$\begin{aligned}\cdots &\to H^{n-k-2}(S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a-d_c)) \to H^{n-k-2}(S^{k}\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-k-2}(S^{k}\widetilde{\Omega}_X(-a)) \to \\ &\to H^{n-k-1}(S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a-d_c)) \to H^{n-k-1}(S^{k}\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-k-1}(S^{k}\widetilde{\Omega}_X(-a)) \to \\ &\to H^{n-k}(S^{k-1}\widetilde{\Omega}_{X_{|X}}(-a-d_c)) \to H^{n-k}(S^{k}\widetilde{\Omega}_{X'_{|X}}(-a)) \to H^{n-k}(S^{k}\widetilde{\Omega}_X(-a)) \to \cdots\end{aligned}$$

Now formula (7.2) tells us that $H^{\ell}(X, S^{k-1}\widetilde{\Omega}_{X'_{|X}}(-a-d_c)) = 0$ for all $\ell \leqslant n-(k-1)-1 = n-k$ and also that $H^{\ell}(X, S^{k}\widetilde{\Omega}_{X'_{|X}}(-a)) = 0$ for all $\ell \leqslant n-k-1$. Combining all this we get the vanishing $H^{\ell}(X, S^{k}\widetilde{\Omega}_X(-a)) = 0$ for all $\ell \leqslant n-k-1$, which is what we wanted.

□

The case $\ell = 0$ is of particular interest to us since we are looking after symmetric differential forms.

Corollary 7.3.5. *With the notation of the theorem, we get*

1. $H^0(X, S^k\widetilde{\Omega}_X(-a)) = 0$ *if* $k < n$.

2. *In particular* $H^0(X, S^k\Omega_X(-a)) = 0$ *if* $k < n$.

Proof. This comes directly from the symmetric powers of the Euler exact sequence

$$0 \to S^k\Omega_X(-a) \to S^k\widetilde{\Omega}_X(-a) \to S^{k-1}\widetilde{\Omega}_X(-a) \to 0.$$

□

Remark 7.3.6. We would like to mention that the second statement of this corollary was already observed by Sakai in [Sak79] (Theorem 8).

Our main interest being the study of $S^k\Omega_X$, it is important to know how to get information on its global sections from global sections of $S^k\widetilde{\Omega}_X$. Corollary 7.3.5 is one result in this direction, but we can be slightly more precise. Take X as before and let

$$\tilde{k}(X, a) := \min\{k \ / \ H^0(X, S^k\widetilde{\Omega}_X(-a)) \neq 0\}$$

and

$$k(X, a) := \min\{k \ / \ H^0(X, S^k\Omega_X(-a)) \neq 0\}$$

Then we have (as in Corollary 7.3.5),

Proposition 7.3.7. *With the above notation,*

1. $k(X, a) = \tilde{k}(X, a) \geqslant n = \dim(X)$

2. $H^0(X, S^{k(X,a)}\Omega_X(-a)) \simeq H^0(X, S^{k(X,a)}\widetilde{\Omega}_X(-a))$.

This tells us that there is one k for which computing the global sections of $S^k\Omega_X$ is the same as computing the global sections of $S^k\widetilde{\Omega}_X$.

7.4 General computation strategy

There are two steps in the computation strategy we will present. We suppose here to simplify that $N = 2c$.

1. Understand what the inclusion $H^0(X, S^{k+c}\widetilde{\Omega}_X(-a)) \subseteq H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D))$ of Corollary 7.3.3 is exactly.

2. For each $\sigma \in H^0(X, S^{k+c}\widetilde{\Omega}_X(-a)) \subseteq H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D))$, using Cech cohomology computations to go back through all the "$\hookrightarrow$" in Corollary 7.3.3, write down a Cech representative for σ.

We were not able up to now to have a full understanding of any of those steps, but we will go through particular examples to illustrate them.

Notation and setting

Before we continue we explain our notation. Let $N \geqslant 2$; we will work in $\mathbb{P}^N$ with homogeneous coordinates $[X_0 : \cdots : X_N]$. As usual we will set $U_i := \{X_i \neq 0\}$. If $X \subseteq \mathbb{P}^N$, we write U_i instead of $X \cap U_i$ if it does not to lead to any confusion. Recall that we have

$$\widetilde{\Omega}_{\mathbb{P}^N} = \bigoplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}^N}(-1)dX_i. \tag{7.3}$$

If $a \geqslant 0$, then

$$H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-a)) \equiv \bigoplus_{\substack{i_0+\cdots+i_N=a \\ i_j>0}} \mathbb{C} \cdot \frac{1}{X_0^{i_0+1} \cdots X_N^{i_N+1}}.$$

We refer to [Har77] Chapter III Theorem 5.1 for a proof of this fact (we also refer to [Ser55]). In particular, we have

$$\begin{aligned} H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)) &= H^N(\mathbb{P}^N, S^k(\bigoplus_{i=0}^{N} \mathbb{C} \cdot dX_i) \otimes \mathcal{O}_{\mathbb{P}^N}(-a-k)) \\ &= \bigoplus_{\substack{|I|=k \\ |J|=a+k-N-1}} \mathbb{C} \cdot \frac{dX^I}{X^{J+\mathbb{I}}}, \end{aligned} \tag{7.4}$$

where as usual I and J denote multi-indices $(i_0, \ldots, i_N)$ and $(j_0, \ldots, j_N)$ and where $dX^I := dX_0^{i_0} \cdots dX_N^{i_N}$ and $X^J := X_0^{j_0} \cdots X_N^{j_N}$. We also set $\mathbb{I} := (1, \ldots, 1)$. Note that $|\mathbb{I}| = N + 1$.

We recall the standard notation concerning Cech cohomology. For $N \geqslant p \geqslant 0$ and $i_0 < \cdots < i_p$, we let

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

If $\mathcal{F}$ is a coherent sheaf on X, we set

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0<\cdot<i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

As usual, the coboundary map d is defined as follows: if $(\sigma_{i_0,\ldots,i_p}) \in C^p(\mathfrak{U}, \mathcal{F})$, we let

$$(d\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \sigma_{i_0,\ldots,\hat{i}_k,\ldots,i_p}{}_{|U_{i_0,\ldots,i_{p+1}}}.$$

Recall that then, one has

$$H^p(X, \mathcal{F}) = \frac{ker\left(C^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{d} C^{p+1}(\mathfrak{U}, \mathcal{F})\right)}{d\left(C^{p-1}(\mathfrak{U}, \mathcal{F})\right)}.$$

Exact sequence in Cech cohomology

Suppose we have an exact sequence

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\pi} \mathcal{G} \to 0.$$

of coherent sheaves on a subvariety $X \subseteq \mathbb{P}^N$. This gives maps between the Cech complexes:

$$\begin{tikzcd}
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] \\
C^{p-1}(\mathfrak{U}, \mathcal{F}) \arrow[r, "\varphi"] \arrow[d, "d"] & C^{p-1}(\mathfrak{U}, \mathcal{E}) \arrow[r, "\pi"] \arrow[d, "d"] & C^{p-1}(\mathfrak{U}, \mathcal{G}) \arrow[d, "d"] \\
C^p(\mathfrak{U}, \mathcal{F}) \arrow[r, "\varphi"] \arrow[d, "d"] & C^p(\mathfrak{U}, \mathcal{E}) \arrow[r, "\pi"] \arrow[d, "d"] & C^p(\mathfrak{U}, \mathcal{G}) \arrow[d, "d"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Suppose that we have the vanishing $H^{p-1}(X,\mathcal{E})=0$. This yields an injection $H^{p-1}(X,\mathcal{G}) \overset{\delta}{\hookrightarrow} H^p(X,\mathcal{F})$. We are now going to describe how one can work with this explicitly in Cech cohomology. That is to say, there are two natural questions:

1. If one has an element $\xi \in H^{p-1}(X,\mathcal{F})$ and a Cech representative $\sigma_{i_0,\dots,i_{p-1}} \in C^{p-1}(\mathfrak{U},\mathcal{G})$ of ξ, how can one write down a cohomology representative of $\delta(\xi)$ in $C^p(\mathfrak{U},\mathcal{F})$?

2. If one has an element $\xi \in H^p(X,\mathcal{F})$ which we know to be in the image of δ_π and a Cech representative $\sigma_{i_0,\dots,i_p} \in C^p(\mathfrak{U},\mathcal{F})$ of ξ, how can one write down a Cech representative of $\delta^{-1}(\xi)$ in $C^{p-1}(\mathfrak{U},\mathcal{G})$?

Those questions will be understood by a diagram chase. For 1), take a cocycle $(\sigma_{i_0,\cdots,i_{p-1}}) \in C^{p-1}(\mathfrak{U},\mathcal{G})$ and suppose that one knows how to extend to a cocycle $(\tilde{\sigma}_{i_0,\dots,i_{p-1}}) \in C^{p-1}(\mathfrak{U},\mathcal{E}))$ (this will always be the case in the upcoming computations). Then compute $d(\tilde{\sigma}_{i_0,\dots,i_{p-1}}) \in C^p(\mathfrak{U},\mathcal{E})$. Since we started with a cocycle, we know that $\pi(d(\tilde{\sigma}_{i_0,\dots,i_{p-1}}))=0$. Therefore there has to be a way of writing $d(\tilde{\sigma}_{i_0,\dots,i_{p-1}})=\varphi(\tau_{i_0,\dots,i_p})$ for some $\tau_{i_0,\dots,i_p} \in C^p(\mathfrak{U},\mathcal{F})$. This τ is the Cech representative we are looking for.

For 2), which is the most interesting question for our purposes, we proceed as follows. Take a cocycle $\sigma_{i_0,\dots,i_p} \in C^p(\mathfrak{U},\mathcal{F})$, then compute $\varphi(\sigma_{i_0,\dots,i_p})$. Then (by our hypothesis) there has to be a way of writing $\varphi(\sigma_{i_0,\dots,i_p})=d(\tau_{i_0,\dots,i_{p-1}})$ for some $(\tau_{i_0,\dots,i_{p-1}}) \in C^{p-1}(\mathfrak{U},\mathcal{E})$ (in our situation this will be done by clearing some power of X_i appearing in the denominators). Then it suffices to compute $\pi(\tau_{i_0,\dots,i_{p-1}})$. This is the Cech representative we are looking for.

We would like to emphasize that we want to compute everything explicitly. Therefore, it is important to see that there are some possible difficulties arising exactly where in the above we wrote “there has to be”: by this, we mean that theoretically we know there is such a expression but it might be very difficult to find it during the actual computations.

The restriction exact sequence in cohomology

Now we describe more precisely what one can say on the restriction exact sequence. Let $F \in \mathbb{C}[X_0,\dots,X_N]$ be a homogeneous degree-e polynomial. We write it $F=\sum_{|K|=e} A_K X^K$. Let $X:=(F=0)$ and let $k,a \geqslant 0$. Then we have the restriction exact sequence tensored by $S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)$:

$$0 \to S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-e) \overset{\cdot F}{\to} S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a) \to S^k\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a) \to 0.$$

This yields a map

$$H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-e)) \overset{\cdot F}{\to} H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)).$$

Under the identification (7.4) we can write this map

$$\begin{array}{ccc} \displaystyle\bigoplus_{\substack{|I|=k \\ |J|=a+e+k-N-1}} \mathbb{C}\cdot \frac{dX^I}{X^{J+\mathbb{I}}} & \to & \displaystyle\bigoplus_{\substack{|I|=k \\ |J|=a+k-N-1}} \mathbb{C}\cdot \frac{dX^I}{X^{J+\mathbb{I}}} \\ \displaystyle\sum_{I,J} \xi_I^J \frac{dX^I}{X^{J+\mathbb{I}}} & \mapsto & \displaystyle\sum_{I,J} \xi_I^J A_K X^{K-J-\mathbb{I}} dX^I, \end{array}$$

where we set $X^{K-J-\mathbb{I}}=0$ if $K \nleq J$. This is because we work in Cech cohomology, and that such an element is indeed 0 in cohomology.

The tilde conormal exact sequence in cohomology

Similarly, let us write explicitly what comes out of the tilde conormal exact sequence. First observe that

$$dF = \sum_{\substack{|K|=e \\ 0\leqslant i\leqslant N}} k_i A_K X^{K-\delta_i} dX_i,$$

where $K := (k_0, \dots, k_N)$ and $\delta_i := (0, \dots, 1, \dots, 0)$, where the only non-zero term appears in the i-th slot.

$$0 \to S^{k-1}\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a-e) \overset{\cdot dF}{\to} S^k\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a) \to S^k\widetilde{\Omega}_X(-a) \to 0.$$

We recall that even though the quotient bundle is defined only over X, the map $\cdot dF$ can be extended over the entire $\mathbb{P}^N$. This is a key point in our computation strategy.

This extended map is just $S^{k-1}\widetilde{\Omega}_{\mathbb{P}^N}(-a-e) \overset{\cdot dF}{\to} S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)$, defined in the straightforward way. It yields a map in cohomology:

$$H^N(\mathbb{P}^N, S^{k-1}\widetilde{\Omega}_{\mathbb{P}^N}(-a-e)) \overset{\cdot dF}{\to} H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)).$$

Under the identification (7.4) we can write it

$$\begin{array}{ccc} \bigoplus\limits_{\substack{|I|=k-1 \\ |J|=a+e+k-N-2}} \mathbb{C}\cdot \dfrac{dX^I}{X^{J+\mathbb{I}}} & \to & \bigoplus\limits_{\substack{|I|=k \\ |J|=a+k-N-1}} \mathbb{C}\cdot \dfrac{dX^I}{X^{J+\mathbb{I}}} \\ \sum\limits_{I,J} \xi_I^J \dfrac{dX^I}{X^{J+\mathbb{I}}} & \mapsto & \sum\limits_{\substack{I,J,K \\ 0\leqslant i\leqslant N}} k_i \xi_I^J A_K X^{K-J-\mathbb{I}-\delta_i} dX^I dX_i. \end{array}$$

As before, if $K-\delta_i \not\leq J$ then $X^{K-J-\mathbb{I}-\delta_i} = 0$.

Interpretation of the H^0

Now we are in a position to give the more precise version of Corollary 7.3.3 we are looking for. If $F \in \mathbb{C}[X_0, \dots, X_N]$ is a homogeneous polynomial of degree e, then, for any $k, a \geqslant 0$, we define:

$$\begin{aligned} K^N_{\cdot F}(k,a) &:= \ker\left(H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)) \overset{\cdot F}{\to} H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a+e))\right), \\ K^N_{\cdot dF}(k,a) &:= \ker\left(H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)) \overset{\cdot dF}{\to} H^N(\mathbb{P}^N, S^{k+1}\widetilde{\Omega}_{\mathbb{P}^N}(-a+e))\right). \end{aligned}$$

If not stated otherwise, we will let $X = H_1 \cap \cdots \cap H_c \subseteq \mathbb{P}^N$ be a complete intersection with $H_i := (F_i = 0)$, where $F_i \in \mathbb{C}[X_0, \dots, X_N]$ is a homogeneous polynomial of degree e_i. When $c = 1$, we let $X := (F = 0)$, where F is a homogeneous degree-e polynomial.

Theorem 7.4.1. *With the above notation, we have an isomorphism (obtained by composing the coboundary maps)*

$$H^{N-2c}(X, S^{k+c}\widetilde{\Omega}_X(-a)) \quad \cong \quad \bigcap_{i=1}^{c} \left(K^N_{\cdot F}(k, a+2D) \cap K^N_{\cdot dF}(k, a+2D)\right).$$

Proof. As usual, we proceed by induction on c. Let $c = 1$. Let $F \in \mathbb{C}[X_0, \dots, X_N]$ be a homogeneous polynomial of degree e.

Using the previous results, we get the commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
H^{N-2}(X, S^{k+1}\widetilde{\Omega}_X(-a)) \arrow[r, "\delta_{\cdot dF}"] \arrow[dr, "\theta_F"] & H^{N-1}(X, S^k\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a-e)) \arrow[r, "\cdot dF"] \arrow[d, "\delta_{\cdot F}"] & H^{N-1}(X, S^{k+1}\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a)) \arrow[d, "\delta_{\cdot F}"] \\
 & H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2e)) \arrow[r, "\cdot dF"] \arrow[d, "\cdot F"] & H^N(\mathbb{P}^N, S^{k+1}\widetilde{\Omega}_{\mathbb{P}^N}(-a-e)) \\
 & H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-e)) \arrow[d] & \\
 & 0 &
\end{tikzcd}
$$

First we are going to prove that the right-hand side square is commutative.

We prove the commutativity by a Cech cohomology computation.
Let $\xi \in \delta_{\cdot F}\left(H^N(X, S^k\widetilde{\Omega}_{\mathbb{P}^N_{|X}}(-a-e))\right) \subseteq H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2e))$. Take $(\sigma_{i_0,\ldots,i_N}) \in C^N(\mathfrak{U}, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2e))$ a Cech representative for ξ. Then $(\sigma_{i_0,\ldots,i_N} \cdot dF)$ is a Cech representative for $\xi \cdot dF$. We have to prove that $\delta_{\cdot F}(\cdot dF(\delta_{\cdot F}^{-1}(\xi))) = \xi \cdot dF$. First we compute $\delta_{\cdot F}^{-1}(\xi)$. To do this we multiply by F to get $(\sigma_{i_0,\ldots,i_N} \cdot F)$. By hypothesis, we know that there is a $(\tau_{i_0,\ldots,i_{N-1}}) \in C^{N-1}(\mathfrak{U}, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-e))$ such that for all $(i_0, \ldots, i_N)$,

$$
d(\tau)_{i_0,\ldots,i_N} = \sum_{k=0}^{N}(-1)^k \tau_{i_0,\ldots,\hat{i}_k,\ldots,i_N}{}_{|U_{i_0,\ldots,i_N}} = \sigma_{i_0,\ldots,i_N} \cdot F.
$$

Then $\delta_{\cdot F}^{-1}((\sigma_{i_0,\ldots,i_N})) = (\tau_{i_0,\ldots,i_{N-1}}{}_{|X})$.

Now we multiply by dF to get $(\tau_{i_0,\cdots,i_{N-1}}{}_{|X} \cdot dF)$, a representative for $\delta_{\cdot F}^{-1}(\xi) \cdot dF$. We just have to apply $\delta_{\cdot F}$ to it. First we extend $(\tau_{i_0,\ldots,i_{N-1}}{}_{|X} \cdot dF)$ to $(\tau_{i_0,\ldots,i_{N-1}} \cdot dF) \in C^{N-1}(\mathbb{P}^N, S^{k+1}\widetilde{\Omega}_{\mathbb{P}^N}(-a))$, then we differentiate it to get

$$
d(\tau \cdot dF)_{i_0,\ldots,i_N} = \sum_{k=0}^{N}(-1)^k \tau_{i_0,\ldots,\hat{i}_k,\ldots,i_N}{}_{|U_{i_0,\ldots,i_N}} \cdot dF = \sigma_{i_0,\ldots,i_N} \cdot F \cdot dF.
$$

Therefore, a Cech representative of $\delta_{\cdot F}(\cdot dF(\delta_{\cdot F}^{-1}(\xi)))$ is $(\sigma_{i_0,\ldots,i_N} \cdot dF)$, which is what we wanted.

Now we are going to prove that $\theta_F(H^{N-2}(X, S^{k+1}\widetilde{\Omega}_X(-a))) = \ker(\cdot F) \cap \ker(\cdot dF)$. Since θ_F is an injection, this will prove our claim.
We have $\theta_F(H^{N-2}(X, S^{k+1}\widetilde{\Omega}_X(-a))) = \delta_{\cdot F}\ker(\cdot dF)$. Now let $\eta \in H^{N-1}(X, S^k\widetilde{\Omega}_{\mathbb{P}^N_X}(-a-e))$. Then,

$$
\begin{aligned}
\eta \in \ker(\cdot dF) \quad &\Leftrightarrow \quad \cdot dF(\eta) = 0 \Leftrightarrow \delta_{\cdot F}(\cdot dF(\eta)) = 0 \\
&\Leftrightarrow \quad \cdot dF(\delta_{\cdot F}(\eta)) = 0 \Leftrightarrow \delta_{\cdot F}(\eta) \in \ker(\cdot dF).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\theta_F(H^{N-2}(X, S^{k+1}\widetilde{\Omega}_X(-a))) &= \ker(\cdot dF) \cap \delta_{\cdot F}(H^{N-1}(X, S^k\widetilde{\Omega}_{\mathbb{P}^N_X}(-a-e))) \\
&= \ker(\cdot F) \cap \ker(\cdot dF).
\end{aligned}
$$

Now let $c > 1$ and suppose the result holds for all $c' < c$. Let $X = X' \cap H_c$ and $X' = H_1 \cap \cdots \cap H_{c-1}$. We observe that, by a similar computation to the above one and by a straightforward induction, the following diagram is commutative:

$$\begin{tikzcd}
H^{N-2c}(X, S^{k+c}\widetilde{\Omega}_X(-a)) \arrow[d, "\delta_{\cdot dF_c}"] & \\
H^{N-2c+1}(X, S^{k+c-1}\widetilde{\Omega}_{X'_{|X}}(-a-e_c)) \arrow[r, "\cdot dF_c"] \arrow[d, "\delta_{\cdot F_c}"] & H^{N-2c+1}(X, S^{k+c}\widetilde{\Omega}_{X'_{|X}}(-a)) \arrow[d, "\delta_{\cdot F_c}"] \\
H^{N-2c+2}(X', S^{k+c-1}\widetilde{\Omega}_{X'}(-a-2e_c)) \arrow[r, "\cdot dF_c"] \arrow[d, "\delta_{\cdot dF_{c-1}}"] & H^{N-2c+2}(X', S^{k+c}\widetilde{\Omega}_{X'}(-a-e_c)) \arrow[d, "\delta_{\cdot dF_{c-1}}"] \\
\vdots \arrow[d, "\delta_{\cdot F_1}"] & \vdots \arrow[d, "\delta_{\cdot F_1}"] \\
H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D)) \arrow[r, "\cdot dF_c"] & H^N(\mathbb{P}^N, S^{k+1}\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D+e_c)).
\end{tikzcd}$$

Similarly, we prove the commutativity of the diagram

$$\begin{tikzcd}
H^{N-2c+1}(X, S^{k+c-1}\widetilde{\Omega}_{X'_{|X}}(-a-e_c)) \arrow[d, "\delta_{\cdot F_c}"] & \\
H^{N-2c+2}(X', S^{k+c-1}\widetilde{\Omega}_{X'}(-a-2e_c)) \arrow[r, "\cdot F_c"] \arrow[d, "\delta_{\cdot dF_{c-1}}"] & H^{N-2c+2}(X', S^{k+c-1}\widetilde{\Omega}_{X'}(-a-e_c)) \arrow[d, "\delta_{\cdot dF_{c-1}}"] \\
\vdots \arrow[d, "\delta_{\cdot F_1}"] & \vdots \arrow[d, "\delta_{\cdot F_1}"] \\
H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D)) \arrow[r, "\cdot F_c"] & H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D+e_c)).
\end{tikzcd}$$

Observe also that all the vertical arrows are injective (this is Corollary 7.3.3). Set $\varphi_c := \delta_{\cdot dF_c} \circ \delta_{\cdot F_c} \circ \cdots \circ \delta_{\cdot F_1}$ and $\varphi_{c-1} := \delta_{\cdot dF_{c-1}} \circ \delta_{\cdot F_{c-1}} \circ \cdots \circ \delta_{\cdot F_1}$. With all this, an easy linear algebra computation proves as above

$$\begin{aligned}
\varphi_c(H^{N-2c}(X, S^{k+c}\widetilde{\Omega}_X(-a))) &= \varphi_{c-1}(H^{N-2(c-1)}(X', S^{k+c-1}\widetilde{\Omega}_X(-a-2e_c))) \\
&\quad \cap \ker(\cdot dF_c) \cap \ker(\cdot F_c)
\end{aligned} \tag{7.5}$$

Then we conclude using the induction hypothesis on $H^{N-2(c-1)}(X', S^{k+c-1}\widetilde{\Omega}_X(-a-2e_c))$.

□

The case $k = 0$ in the above theorem can be simplified using Euler's Formula for polynomials in several variables, this is the content of the following result.

Theorem 7.4.2. *With the above notation, we have an isomorphism (obtained by composing the coboundary maps)*

$$\begin{aligned}
H^{N-2c}(X, S^c\Omega_X(-m)) &= H^{N-2c}(X, S^c\widetilde{\Omega}_X(-m)) \\
&\cong \bigcap_{i=1}^{c} \ker\left(H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m-2D)) \overset{\cdot dF_i}{\to} H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m-2D+d_i))\right).
\end{aligned}$$

Proof. We will prove that for all $a \geqslant 0$ and for any degree-e homogeneous polynomial F, one has

$$K^N_{\cdot dF}(0,a) \subseteq K^N_{\cdot F}(0,a).$$

This is based on the fact that

$$\sum_{i=0}^{N} X_i \frac{\partial F}{\partial X_i} = eF.$$

Let $\xi \in K^N_{\cdot dF}(0,a)$. We see that $\xi \cdot dF = \sum_{i=0}^{N} \xi \cdot \frac{\partial F}{\partial X_i} dX_i = 0$ if and only if $\xi \cdot \frac{\partial F}{\partial X_i} = 0$ for all $0 \leqslant i \leqslant N$. Therefore, if $\xi \cdot dF = 0$, then

$$\begin{aligned} \xi \cdot F &= \frac{1}{e} \xi \sum_{i=0}^{N} X_i \frac{\partial F}{\partial X_i} \\ &= \frac{1}{e} \sum_{i=0}^{N} X_i \xi \frac{\partial F}{\partial X_i} = 0 \end{aligned}$$

(note that all those computations take place in an H^N). This concludes the proof. $\square$

Sharpness in Corollary 7.3.5**.**

Using Theorem 7.4.2, we will see that Corollary 7.3.5 is sharp. More precisely, for each $n \geqslant 1$, we construct an example of an n-dimensional smooth complete intersection variety X satisfying $H^0(X, S^n\Omega_X) \simeq H^0(X, S^n\widetilde{\Omega}_X) \neq 0$. Fix $n \geqslant 1$, and take an $n \times (2n+1)$ matrix

$$\begin{pmatrix} a_{1,0} & \cdots & a_{1,2n} \\ \vdots & & \vdots \\ a_{n,0} & \cdots & a_{n,2n} \end{pmatrix}$$

such that none of its $(n \times n)$-minors vanishes. For each $0 \leqslant i \leqslant n$, consider the following Fermat-like homogeneous degree-e polynomial in $2n+1$ variables

$$F_i := \sum_{k=0}^{2n} a_{i,k} X_k^e.$$

Let $X := \bigcap_{i=1}^{n} (F_i = 0) \subseteq \mathbb{P}^{2n}$. The condition we have on our matrix M ensures, by the Jacobian criterion, that X is a smooth complete intersection variety. We know that

$$H^0(X, S^c\Omega_X) = \bigcap_{i=1}^{c} \ker \left(H^{2n}(\mathbb{P}^{2n}, \mathcal{O}_{\mathbb{P}^{2n}}(-2ne)) \xrightarrow{\cdot dF_i} H^{2n}(\mathbb{P}^{2n}, \widetilde{\Omega}_{\mathbb{P}^{2n}}(-(2n-1)e)) \right).$$

Recall that

$$H^{2n}(\mathbb{P}^{2n}, \mathcal{O}_{\mathbb{P}^{2n}}(-2ne)) = \bigoplus_{|I| := 2ne-2n-1} \mathbb{C} \cdot \frac{1}{X^{I+\mathbb{I}}}.$$

Observe that if $I = (i_0, \dots, i_N)$, with $|I| = 2ne - 2n - 1$, is a multi-index satisfying $i_k \leqslant e-2$ for any $0 \leqslant k \leqslant 2n$, then

$$\forall\ 1 \leqslant \ell \leqslant n \quad \frac{1}{X^{I+\mathbb{I}}} \cdot dF_\ell \equiv 0 \in H^{2n}(\mathbb{P}^{2n}, \widetilde{\Omega}_{\mathbb{P}^{2n}}(-(2n-1)e)),$$

which proves that the kernel is non-zero as soon as $2ne \leqslant (2n+1)(e-1)$. This gives the desired example as soon as $e \geqslant 2n+1$. We will study this example more thoroughly in the case $n = 2$.

The Euler condition.

There is one issue that we would like to detail somewhat more. We would like to know when a section $\tilde{\sigma} \in H^0(X, S^k\widetilde{\Omega}_X(-a))$ is actually a section $\sigma \in H^0(X, S^k\Omega_X(-a))$. This leads us to another version of Theorem 7.4.1. Let us set

$$E^N(k,a) := \ker(H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a)) \to H^N(\mathbb{P}^N, S^{k-1}\widetilde{\Omega}_{\mathbb{P}^N}(-a))).$$

Note that this is just $H^N(\mathbb{P}^N, S^k\Omega_{\mathbb{P}^N}(-a))$ embedded in $H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a))$. This is an irreducible representation of $GL_{N+1}(\mathbb{C})$.

Theorem 7.4.3. *With the notation of Theorem* 7.4.1*, we have an isomorphism*

$$H^{N-2c}(X, S^{k+c}\Omega_X(-a)) \quad \cong \quad E^N(k, a+2D) \cap \bigcap_{i=1}^{c} \left(K^N_{\cdot F}(k, a+2D) \cap K^N_{\cdot dF}(k, a+2D)\right).$$

Proof. The proof is very similar to what was done above. It is reduced to proving the commutativity of the following diagram

$$\begin{array}{ccc}
H^{N-2c}(X, S^{k+c}\Omega_X(-a)) & & \\
\downarrow & & \\
H^{N-2c}(X, S^{k+c}\widetilde{\Omega}_X(-a)) & \xrightarrow{\zeta} & H^{N-2c}(X, S^{k+c-1}\widetilde{\Omega}_X(-a)) \\
\downarrow{\scriptstyle \delta_{\cdot dF_c}} & & \downarrow{\scriptstyle \delta_{\cdot dF_c}} \\
H^{N-2c+1}(X, S^{k+c-1}\widetilde{\Omega}_{X'_{|X}}(-a-e_c)) & \xrightarrow{\zeta} & H^{N-2c+1}(X, S^{k+c-2}\widetilde{\Omega}_{X'_{|X}}(-a-e_c)) \\
\downarrow{\scriptstyle \delta_{\cdot F_c}} & & \downarrow{\scriptstyle \delta_{\cdot F_c}} \\
\vdots & & \vdots \\
\downarrow{\scriptstyle \delta_{\cdot F_1}} & & \downarrow{\scriptstyle \delta_{\cdot F_1}} \\
H^N(\mathbb{P}^N, S^k\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D)) & \xrightarrow{\zeta} & H^N(\mathbb{P}^N, S^{k-1}\widetilde{\Omega}_{\mathbb{P}^N}(-a-2D)),
\end{array}$$

where all the horizontal arrows are the ones arising from the Euler exact sequence. This is done as in the proof of Theorem 7.4.1. ☐

7.5 Complete intersection surfaces revisited

Now we will focus (again) on complete intersection surfaces and in particular on complete intersection surfaces in $\mathbb{P}^4$. We already proved that a generic complete intersection surface of high multidegree in $\mathbb{P}^4$ has ample cotangent bundle. To do this proof we used holomorphic Morse inequalities to construct symmetric differential forms on such surfaces. However the use of holomorphic Morse inequalities has two drawbacks:

1. We don't know up to which k one has to go to get a nonvanishing for $H^0(X, S^k\Omega_X)$.

2. We don't have any explicit equations for such a symmetric differential form.

Let us explain the content of this section. Bogomolov [Bog78] proved some nonvanishing results concerning symmetric differential forms on some surfaces with $c_1^2 - c_2 > 0$, using Riemann-Roch computations. In that same article, he constructs an example of a simply connected surface S such that $H^0(S, S^2\Omega_S) \neq 0$. In [BDO08], Bogomolov and De Oliveira, among other things, construct an example of a family of surfaces for which the dimension $h^0(S_t, S^m\Omega_{S_t})$ jumps as t varies, illustrating the fact that these numbers are not deformation-invariant. We start by redoing Bogomolov's Riemann-Roch computation for complete intersection surfaces. These computations allow us to give explicit bounds for k (with the notation of problem 1)). Then we illustrate our computation strategy by constructing a family of complete intersection surfaces S_δ^γ such that $H^0(S_0^0, S^2\Omega_{S_0^0}) \neq 0$ and for generic $(\delta,\gamma) \in (\mathbb{C}^2)^2$, $H^0(S_\delta^\gamma, S^2\Omega_{S_\delta^\gamma}) = 0$, giving a new example of the non-invariance under deformation of $h^0(X, S^2\Omega_X)$. Moreover, we make all the computations for the special fibre S_0^0, describing explicitly all the elements of $H^0(S_0^0, S^2\Omega_{S_0^0})$. This is a first illustration on how one can try to solve problem 2).

7.5.1 Riemann-Roch computations

In [Bog78], Bogomolov stated a non-vanishing theorem for a complete intersection surface. However he did not write down the computations explicitly. Moreover, as such a result is only available for surfaces, it emphasizes the peculiarities of the surface case. For both these reasons we will reprove it here, but it must be clear that there is nothing in this section that is new.

Recall that if E is a rank-2 vector bundle on a variety and if $k \geqslant 1$ then:

$$\begin{aligned} c_1(S^kE) &= \binom{k+1}{2} c_1(E), \\ c_2(S^kE) &= \frac{3k+2}{4}\binom{k+1}{3} c_1(E)^2 + \binom{k+2}{3} c_2. \end{aligned}$$

We will need the Hirzebruch-Riemann-Roch Theorem for vector bundles on surfaces (we refer to [Ful98] Example 15.2.2).

Hirzebruch-Riemmann-Roch Theorem. *Let X be a smooth complete surface. Let $c_i := c_i(TX)$. Then,*

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}\int_X c_1^2 + c_2.$$

If E is a rank-e vector bundle on X with $e_i := c_i(E)$, then

$$\chi(X, E) = \frac{1}{2}\int_X (e_1^2 - 2e_2 + c_1e_1) + e\chi(X, \mathcal{O}_X).$$

As a particular case, if E is a rank-2 vector bundle on a surface, with Chern classes $e_i := c_i(E)$, then:

$$\chi(X, S^kE) = (k+1)\chi(X, \mathcal{O}_X) + \int_X \frac{2k+1}{6}\binom{k+1}{2} e_1^2 - \binom{k+2}{3} e_2 + \frac{1}{2}\binom{k+1}{2} e_1c_1.$$

Remark 7.5.1. Bogomolov [Bog78] then proves that if $\ell \det(E) + K_X$ is effective for $\ell \gg 0$ and if $e_1^2 > e_2$ then $h^0(X, S^kE) \neq 0$ for $k \gg 0$. His argument goes as follows. $\chi(X, S^k\Omega_X)$ is a degree-3 polynomial when viewed as a polynomial in k, whose dominant term is $\frac{c_1^2-c_2}{6}k^3$. Therefore, if $c_1^2 > c_2$ and k is big enough, we get $\chi(X, S^k\Omega_X) > 0$. Thus $h^0(X, S^kE) + h^2(X, S^kE) > 0$ for $k \gg 0$. Then he concludes noticing that

$$h^2(X, S^kE) = h^0(X, S^kE^* \otimes K_X) = h^0(X, S^kE \otimes K_X \otimes \det(E)^{-k}) \leqslant h^0(X, S^kE).$$

Now we apply all this to complete intersection surfaces in $\mathbb{P}^N$, to get a estimate on k and on the multidegree of the surfaces to achieve such a nonvanishing. We take our usual notation: $X = H_1 \cap \cdots \cap H_{N-2} \subset \mathbb{P}^N$ where $N \geqslant 4$ and where H_i is a degree-d_i hypersurface for each $1 \leqslant i \leqslant N-2$. Moreover, we let $c_i := c_i(TX)$. From computations just as the ones in section 3.2, it comes that

$$\begin{aligned} c_2 &= c_2(\Omega_X) = \sum_i d_i^2 + \sum_{i<j} d_id_j - (N+1)\sum_i d_i + \binom{N+1}{2}, \\ c_1 &= -c_1(\Omega_X) = -\sum_i d_i + (N+1). \end{aligned}$$

Therefore,

$$c_1^2 = \sum_i d_i^2 + 2\sum_{i<j} d_i d_j - 2(N+1)\sum_i d_i + (N+1)^2.$$

Remark 7.5.2. At this point we would like to mention that $c_1^2 - 2c_2 = -\sum_i d_i^2 + (N+1) < 0$, hence these surfaces are not of positive index. Thus one cannot apply Miyaoka's result [Miy83], which states that a surface with positive index has almost everywhere ample cotangent bundle.

We now apply the above formula to $S^k\Omega_X$:

$$\begin{aligned}
\chi(X, S^k\Omega_X) &= \frac{(k+1)}{12}\int_X \left((2k^2-2k+1)c_1^2 + (1-4k-2k^2)c_2\right) \\
&= \frac{(k+1)}{12} d_1 \cdots d_{N-2} \left((2-6k)\sum_i d_i^2 + (N+1)(-2k^2+8k-3)\sum_i d_i \right. \\
&\qquad \left. +(2k^2-8k+3)\sum_{i<j} d_i d_j + \frac{N+1}{2}(2k^2(N+2) + (2N+1)(1-4k))\right).
\end{aligned}$$

From now on, we suppose that $d_1 = \cdots = d_{N-2} = d$, so that as a polynomial in d, the leading term in this formula is

$$\frac{k+1}{12}\left((2-6k)(N-2) + \binom{N-2}{2}(2k^2-8k+3)\right) d^N.$$

Looking precisely at when that polynomial turns out to be positive, we find that:

$$\begin{aligned}
\text{for N} = 4\ ,&\ \ h^0(X, S^k\Omega_X) \neq 0 \ \text{ as soon as }\ k \geqslant 10 \text{ and } d \gg 1, \\
\text{for N} = 5\ ,&\ \ h^0(X, S^k\Omega_X) \neq 0 \ \text{ as soon as }\ k \geqslant 7 \text{ and } d \gg 1, \\
\text{for } 6 \leqslant \text{N} \leqslant 7\ ,&\ \ h^0(X, S^k\Omega_X) \neq 0 \ \text{ as soon as }\ k \geqslant 6 \text{ and } d \gg 1, \\
\text{for } 8 \leqslant \text{N} \leqslant 17\ ,&\ \ h^0(X, S^k\Omega_X) \neq 0 \ \text{ as soon as }\ k \geqslant 5 \text{ and } d \gg 1, \\
\text{for } 18 \leqslant \text{N}\ ,&\ \ h^0(X, S^k\Omega_X) \neq 0 \ \text{ as soon as }\ k \geqslant 4 \text{ and } d \gg 1.
\end{aligned}$$

7.5.2 A Fermat-type complete intersection in $\mathbb{P}^4$

Now we make explicitly all the computations in the case of the intersection of two Fermat-type hypersurfaces in $\mathbb{P}^4$. We will see that some unexpected behaviours appear. More precisely, fix $a \in \mathbb{N}$ and let $e \in \mathbb{N}$, such that $e \geqslant 5 + a$. Let also

$$\begin{aligned}
F &:= X_0^e + X_1^e + X_2^e + X_3^e + X_4^e, \\
G &:= a_0X_0^e + a_1X_1^e + a_2X_2^e + a_3X_3^e + a_4X_4^e,
\end{aligned}$$

where $a_0, \cdots, a_4 \in \mathbb{C}$ are such that all the $(2,2)$-minors $a_{ij} = a_i - a_j$ of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

are non zero (this ensures that our intersection is smooth). Now we let $H_F := (F = 0)$, $H_G := (G = 0)$, and $X := H_F \cap H_G$. Our first goal is to compute explicitly (in Cech cohomology) $H^0(X, S^2\Omega_X(-a)) = H^0(X, S^2\widetilde{\Omega}_X(-a))$. We will show that this group is non-zero and afterwards we will also show that there are some deformations of this surface for which this group is zero.

First we compute the partial derivatives.

$$\begin{aligned} \frac{\partial F}{\partial X_i} &= eX_i^{e-1}, \\ \frac{\partial G}{\partial X_i} &= ea_iX_i^{e-1}. \end{aligned}$$

It is therefore easy to see that with the notation of Theorem 7.4.2, the kernel $K^4_{\cdot dF}(0,a) \cap K^4_{\cdot dG}(0,a)$ is spanned by the elements of the form

$$\frac{1}{X_0^{e-1-\epsilon_0}X_1^{e-1-\epsilon_1}X_2^{e-1-\epsilon_2}X_3^{e-1-\epsilon_3}X_4^{e-1-\epsilon_4}},$$

where $0 \leq \epsilon_k \leqslant e-2$ and $5e-5-\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4 = 4e+a$, or equivalently $|\epsilon| = e-5-a \geqslant 0$. Now, to have some information on the base locus of these sections, we will backtrack through all the long exact sequences mentioned above, by looking carefully at what happens in Cech cohomology.

Fix a suitable ϵ. Let

$$\sigma^{\epsilon}_{01234} := \frac{1}{X_0^{e-1-\epsilon_0}X_1^{e-1-\epsilon_1}X_2^{e-1-\epsilon_2}X_3^{e-1-\epsilon_3}X_4^{e-1-\epsilon_4}}.$$

This is an element of $H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-4e-a))$.

We have the chain of inclusions

$$\begin{aligned} H^0(X, S^2\widetilde{\Omega}_X(-a)) &\hookrightarrow H^1(X, \widetilde{\Omega}_{H_F|X}(-e-a)) \\ &\hookrightarrow H^2(H_F, \widetilde{\Omega}_{H_F}(-2e-a)) \\ &\hookrightarrow H^3(H_F, \mathcal{O}_{H_F}(-3e-a)) \\ &\hookrightarrow H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-4e-a)). \end{aligned}$$

Representative in the H^3

We know already that $\sigma^{\epsilon}_{01234}$ lies in $H^3(H_F, \mathcal{O}_{H_F}(-3e-a))$, viewed as a subspace of $H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-4e-a))$. We will now compute a Cech cohomology representative of this section when viewed in the H^3.

$$\begin{aligned} \sigma^{\epsilon}_{01234} \cdot F &= \frac{X_0^{1+\epsilon_0}}{X_1^{e-1-\epsilon_1}X_2^{e-1-\epsilon_2}X_3^{e-1-\epsilon_3}X_4^{e-1-\epsilon_4}} \\ &+\frac{X_1^{1+\epsilon_1}}{X_0^{e-1-\epsilon_0}X_2^{e-1-\epsilon_2}X_3^{e-1-\epsilon_3}X_4^{e-1-\epsilon_4}} \\ &+\frac{X_2^{1+\epsilon_2}}{X_0^{e-1-\epsilon_0}X_1^{e-1-\epsilon_1}X_3^{e-1-\epsilon_3}X_4^{e-1-\epsilon_4}} \\ &+\frac{X_3^{1+\epsilon_3}}{X_0^{e-1-\epsilon_0}X_1^{e-1-\epsilon_1}X_2^{e-1-\epsilon_2}X_4^{e-1-\epsilon_4}} \\ &+\frac{X_4^{1+\epsilon_4}}{X_0^{e-1-\epsilon_0}X_1^{e-1-\epsilon_1}X_2^{e-1-\epsilon_2}X_3^{e-1-\epsilon_3}}. \end{aligned}$$

Therefore we can split this expression according to the value of the denominator to get a Cech cocycle $(\sigma^\epsilon_{1234}, \sigma^\epsilon_{0234}, \sigma^\epsilon_{0134}, \sigma^\epsilon_{0124}, \sigma^\epsilon_{0123})$, where

$$\begin{aligned}
\sigma^\epsilon_{0123} &= \frac{X_4^{1+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}}, \\
\sigma^\epsilon_{0124} &= \frac{-X_3^{1+\epsilon_3}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_4^{e-1-\epsilon_4}}, \\
\sigma^\epsilon_{0134} &= \frac{X_2^{1+\epsilon_2}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_3^{e-1-\epsilon_3} X_4^{e-1-\epsilon_4}}, \\
\sigma^\epsilon_{0234} &= \frac{-X_1^{1+\epsilon_1}}{X_0^{e-1-\epsilon_0} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3} X_4^{e-1-\epsilon_4}}, \\
\sigma^\epsilon_{1234} &= \frac{X_0^{1+\epsilon_0}}{X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3} X_4^{e-1-\epsilon_4}}.
\end{aligned}$$

Observe that this is what we are looking for, since $d(\sigma^\epsilon_{ijk\ell}) = \sigma^\epsilon_{01234} \cdot F$.

Representative in the H^2

From this, we are able to deduce the representative in the H^2. The principle is the same: we multiply by dF and then we try to clear some parts of the denominators to find the desired cocycle. We will see that it is not as straightforward as previously, and we will need to use the relation $F = 0$.

$$\begin{aligned}
\sigma^\epsilon_{0123} \cdot dF = &\frac{eX_0^{\epsilon_0} X_4^{1+\epsilon_4}}{X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} dX_0 \\
&+ \frac{eX_1^{\epsilon_1} X_4^{1+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} dX_1 \\
&+ \frac{eX_2^{\epsilon_2} X_4^{1+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_3^{e-1-\epsilon_3}} dX_2 \\
&+ \frac{eX_3^{\epsilon_3} X_4^{1+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2}} dX_3 \\
&+ \frac{X_4^{e+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} dX_4.
\end{aligned}$$

The last term cannot be directly seen as the image of an element under the Cech differential, and we need to use the relation $F = 0$. We get therefore $X_4^e = -X_0^e - X_1^e - X_2^e - X_3^e$.

$$\begin{aligned}
\frac{X_4^{e+\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} = &\frac{-eX_0^{1+\epsilon_0} X_4^{\epsilon_4}}{X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} \\
&+ \frac{-eX_1^{1+\epsilon_1} X_4^{\epsilon_4}}{X_0^{e-1-\epsilon_0} X_2^{e-1-\epsilon_2} X_3^{e-1-\epsilon_3}} \\
&+ \frac{-eX_2^{1+\epsilon_2} X_4^{\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_3^{e-1-\epsilon_3}} \\
&+ \frac{-eX_3^{1+\epsilon_3} X_4^{\epsilon_4}}{X_0^{e-1-\epsilon_0} X_1^{e-1-\epsilon_1} X_2^{e-1-\epsilon_2}}.
\end{aligned}$$

We see that

$$\sigma^\epsilon_{0123} \cdot dF = \sigma^\epsilon_{123} - \sigma^\epsilon_{023} + \sigma^\epsilon_{013} - \sigma^\epsilon_{012},$$

where

$$\sigma^{\epsilon}_{\hat{i}\hat{j}} \quad = \quad \frac{(-1)^{i+j} e (X_i^{\epsilon_i} X_j^{1+\epsilon_j} dX_i - X_i^{1+\epsilon_i} X_j^{\epsilon_j} dX_j)}{X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_4^{e-1-\epsilon_4}}.$$

By doing the same computations with $\sigma^{\epsilon}_{0124}, \sigma^{\epsilon}_{0134}, \sigma^{\epsilon}_{0234}$ and σ^{ϵ}_{1234} we see that with the above formula,

$$(\sigma^{\epsilon}_{ijk\ell}) \cdot dF = d(\sigma^{\epsilon}_{ijk}),$$

where d denotes the Cech differential.

Representative in the H^1

Now we just have to multiply the obtained cocycle by G, then simplify it using F and dF to get a cocycle representative of the class in H^1:

$$\begin{aligned}
\sigma^{\epsilon}_{\hat{i}\hat{j}} \cdot G \quad &= \quad \frac{(-1)^{i+j} e (X_i^{\epsilon_i} X_j^{1+\epsilon_j} dX_i - X_i^{1+\epsilon_i} X_j^{\epsilon_j} dX_j)}{X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_4^{e-1-\epsilon_4}} \cdot G \\
&= \quad \frac{(-1)^{i+j} e (X_i^{\epsilon_i} X_j^{1+\epsilon_j} dX_i - X_i^{1+\epsilon_i} X_j^{\epsilon_j} dX_j)}{X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_4^{e-1-\epsilon_4}} \cdot \sum_{k \neq i,j} a_k X_k^e \\
&+ \quad \frac{(-1)^{i+j} e (X_i^{\epsilon_i} X_j^{1+\epsilon_j} dX_i - X_i^{1+\epsilon_i} X_j^{\epsilon_j} dX_j)}{X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_4^{e-1-\epsilon_4}} \cdot a_i X_i^e \\
&+ \quad \frac{(-1)^{i+j} e (X_i^{\epsilon_i} X_j^{1+\epsilon_j} dX_i - X_i^{1+\epsilon_i} X_j^{\epsilon_j} dX_j)}{X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_4^{e-1-\epsilon_4}} \cdot a_j X_j^e.
\end{aligned}$$

Now we observe that using the relations $-X_i^e = \sum_{k \neq i} X_k^e$ and $X_i^{e-1} dX_i + X_j^{e-1} dX_j = -\sum_{k \neq i,j} X_k^{e-1} dX_k$, we get

$$\begin{aligned}
(X_j dX_i - X_i dX_j) \cdot X_i^e \quad &= \quad X_i \left(X_j X_i^{e-1} dX_i - X_i^e dX_j \right) \\
&= \quad X_i \left(X_j X_i^{e-1} dX_i + \sum_{k \neq i} X_k^e dX_j \right) \\
&= \quad X_i X_j \left(X_i^{e-1} dX_i + X_j^{e-1} dX_j \right) + X_i \sum_{k \neq i,j} X_k^e dX_j \\
&= \quad X_i \sum_{k \neq i,j} X_k^{e-1} (X_k dX_j - X_j dX_k).
\end{aligned}$$

Similarly,

$$(X_j dX_i - X_i dX_j) \cdot X_j^e \quad = \quad X_j \sum_{k \neq i,j} X_k^{e-1} (X_i dX_k - X_k dX_i).$$

Combining all this together we obtain,

$$\sigma^{\epsilon}_{\hat{i}\hat{j}} \cdot G = (-1)^{i+j} \sum_{k \neq i,j} X^{\epsilon}_{ijk} \frac{(X_j X_k (a_k - a_j) dX_i + X_i X_k (a_i - a_k) dX_j + X_i X_j (a_j - a_i) dX_k)}{X^{e-1-\epsilon}_{\hat{i}\hat{j}\hat{k}}},$$

where $X^{\epsilon}_{ijk} = X_i^{\epsilon_i} X_j^{\epsilon_j} X_k^{\epsilon_k}$ and $X^{e-1-\epsilon}_{\hat{i}\hat{j}\hat{k}} = X_0^{e-1-\epsilon_0} \cdots \hat{X}_i \cdots \hat{X}_j \cdots \hat{X}_k \cdots X_4^{e-1-\epsilon_4}$. Therefore, we see that $(\sigma_{ijk}) \cdot G = d(\sigma_{ij})$, where

$$\sigma^{\epsilon}_{ij} = (-1)^{i+j} X^{\epsilon}_{k\ell m} \frac{(X_k X_m (a_k - a_m) dX_\ell + X_k X_\ell (a_\ell - a_k) dX_m + X_\ell X_m (a_m - a_\ell) dX_k)}{X_i^{e-1-\epsilon_i} X_j^{e-1-\epsilon_j}}$$

for $k < \ell < m$ and $\{i, j, k, \ell, m\} = \{0, 1, 2, 3, 4\}$.

Representative in the H^0

To find the representative in H^0, we multiply by dG and simplify using the other relations. This is the trickiest part of all these computations. Let again $k < \ell < m$ and $i < j$ such that $\{i, j, k, \ell, m\} = \{0, 1, 2, 3, 4\}$. We have:

$$\begin{aligned}\sigma^{\epsilon}_{ij} \cdot dG &= \sigma^{\epsilon}_{ij} \cdot (a_i X_i^{e-1} dX_i + a_j X_j^{e-1} dX_j) \\ &\quad + \sigma^{\epsilon}_{ij} \cdot (a_k X_k^{e-1} dX_k + a_\ell X_\ell^{e-1} dX_\ell + a_m X_m^{e-1} dX_m).\end{aligned}$$

We have to simplify the second member of the right-hand side. First we note that we have the relations

$$\begin{aligned}0 &= G - a_m F = \sum_{0 \leqslant p \leqslant 4} (a_p - a_m) X_p^e, \\ 0 &= a_\ell F - G = \sum_{0 \leqslant p \leqslant 4} (a_\ell - a_p) X_p^e.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(a_k - a_m) X_k^e &= -(a_i - a_m) X_i^e - (a_j - a_m) X_j^e - (a_\ell - a_m) X_\ell^e, \\ (a_\ell - a_k) X_k^e &= -(a_\ell - a_i) X_i^e - (a_\ell - a_j) X_j^e - (a_\ell - a_m) X_m^e.\end{aligned}$$

Note also that using the relation $dF = 0$, we get

$$X_i^{e-1} dX_i + X_j^{e-1} dX_j = -X_k^{e-1} dX_k - X_\ell^{e-1} dX_\ell - X_m^{e-1} dX_m.$$

We can use this to simplify the right-hand side as follows:

$$\begin{aligned}&((a_k - a_m) X_k X_m dX_\ell + (a_\ell - a_k) X_k X_\ell dX_m + (a_m - a_\ell) X_\ell X_m dX_k) \cdot X_k^{e-1} \\ &= ((a_k - a_m) X_k^e X_m dX_\ell + (a_\ell - a_k) X_k^e X_\ell dX_m + (a_m - a_\ell) X_k^{e-1} X_\ell X_m dX_k) \\ &= (-(a_i - a_m) X_i^e X_m dX_\ell - (a_\ell - a_i) X_i^e X_\ell dX_m \\ &\quad -(a_j - a_m) X_j^e X_m dX_\ell - (a_\ell - a_j) X_j^e X_\ell dX_m \\ &\quad -(a_\ell - a_m) X_\ell^e X_m dX_\ell - (a_\ell - a_m) X_m^e X_\ell dX_m - (a_\ell - a_m) X_k^{e-1} X_\ell X_m dX_m) \\ &= (-(a_i - a_m) X_i^e X_m dX_\ell - (a_\ell - a_i) X_i^e X_\ell dX_m \\ &\quad -(a_j - a_m) X_j^e X_m dX_\ell - (a_\ell - a_j) X_j^e X_\ell dX_m \\ &\quad +(a_\ell - a_m) X_\ell^e X_m (X_i^{e-1} dX_i + X_j^{e-1} dX_j) \\ &= X_i^{e-1}((a_m - a_i) X_m X_i dX_\ell + (a_i - a_\ell) X_\ell X_i dX_m + (a_\ell - a_m) X_\ell X_m dX_i) \\ &\quad + X_j^{e-1}((a_m - a_j) X_m X_j dX_\ell + (a_j - a_\ell) X_\ell X_j dX_m + (a_\ell - a_m) X_\ell X_m dX_j).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&((a_k - a_m) X_k X_m dX_\ell + (a_\ell - a_k) X_k X_\ell dX_m + (a_m - a_\ell) X_\ell X_m dX_k) \cdot X_\ell^{e-1} \\ &= X_i^{e-1}((a_k - a_i) X_k X_i dX_m + (a_i - a_m) X_m X_i dX_k + (a_m - a_k) X_m X_k dX_i) \\ &\quad + X_j^{e-1}((a_k - a_j) X_k X_j dX_m + (a_j - a_m) X_m X_j dX_k + (a_m - a_k) X_m X_k dX_j)\end{aligned}$$

and also

$$\begin{aligned}&((a_k - a_m) X_k X_m dX_\ell + (a_\ell - a_k) X_k X_\ell dX_m + (a_m - a_\ell) X_\ell X_m dX_k) \cdot X_m^{e-1} \\ &= X_i^{e-1}((a_i - a_k) X_i X_k dX_\ell + (a_\ell - a_i) X_\ell X_m dX_k + (a_k - a_\ell) X_\ell X_k dX_i) \\ &\quad + X_j^{e-1}((a_j - a_k) X_j X_k dX_\ell + (a_\ell - a_j) X_\ell X_j dX_k + (a_k - a_\ell) X_\ell X_k dX_j).\end{aligned}$$

Expanding everything, one can simplify in each denominator either $X_i^{e-1-\epsilon_1}$ or $X_j^{e-1-\epsilon_j}$. Eventually, one obtains $\sigma_{ij}^\epsilon \cdot dG = \sigma_j^\epsilon - \sigma_i^\epsilon$, where

$$\begin{aligned}\sigma_j^\epsilon &= (-1)^j \frac{X_{ik\ell m}^\epsilon}{X_j^{e-1-\epsilon_j}}(a_{ki\ell m}X_\ell X_m dX_i dX_k + a_{kmi\ell}X_m X_k dX_i dX_\ell + a_{mik\ell}X_k X_\ell dX_i dX_m \\ &\quad + a_{ki\ell m}X_i X_k dX_\ell dX_m + a_{kmi\ell}X_\ell X_i dX_k dX_m + a_{mik\ell}X_i X_m dX_k dX_\ell),\end{aligned}$$

where $X_{ik\ell m}^\epsilon = X_i^{\epsilon_i}X_j^{\epsilon_j}X_\ell^{\epsilon_\ell}X_m^{\epsilon_m}$, and where $a_{pqrs} = (a_p - a_q)(a_r - a_s)$. One can observe the relation $a_{ki\ell m} + a_{kmi\ell} + a_{mik\ell} = 0$. This gives us, at last, the desired representative in H^0.

Representative as a symmetric differential form

As we said before, this is not only a section in $H^0(S, S^2\widetilde{\Omega}_S(-m))$: it is in fact a section in $H^0(S, S^2\Omega_S(-m))$. Again, let us write down as explicitly as possible the obtained section in local charts. We work on U_0 so that we can take coordinates $x_i := \frac{X_i}{X_0}$ such that $dx_i = \frac{X_0 dX_i - X_i dX_0}{X_0^2}$.

The equation of $S_0 := S \cap U_0$ becomes

$$\begin{aligned}f &:= 1 + x_1^e + x_2^e + x_3^e + x_4^e, \\ g &:= a_0 + a_1x_1^e + a_2x_2^e + a_3x_3^e + a_4x_4^e,\end{aligned}$$

and the differentials

$$\begin{aligned}df &:= e(x_1^{e-1}dx_1 + x_2^{e-1}dx_2 + x_3^{e-1}dx_3 + x_4^{e-1}dx_4), \\ dg &:= e(a_1x_1^{e-1}dx_1 + a_2x_2^{e-1}dx_2 + a_3x_3^{e-1}dx_3 + a_4x_4^{e-1}dx_4).\end{aligned}$$

We look at the first Jacobian

$$J_{1,2}(f,g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} = e^2(a_1 - a_0)(x_1x_2)^{e-1}.$$

We now restrict ourselves to the open subset $(J_{1,2}(f,g) \neq 0)$, which is just $(x_1 \neq 0) \cap (x_2 \neq 0)$. Let us set $a_{pq} = a_p - a_q$. It is now a matter of simple linear algebra to obtain

$$\begin{aligned}x_1^e &= \frac{1}{a_{21}}(a_{02} + a_{32}x_3^e + a_{42}x_4^e), \\ x_2^e &= \frac{1}{a_{21}}(a_{10} + a_{13}x_3^e + a_{14}x_4^e).,\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}dx_1 &= \frac{a_{32}}{a_{21}}\left(\frac{x_3}{x_1}\right)^{e-1}dx_3 + \frac{a_{42}}{a_{21}}\left(\frac{x_4}{x_1}\right)^{e-1}dx_4, \\ dx_2 &= \frac{a_{13}}{a_{21}}\left(\frac{x_3}{x_1}\right)^{e-1}dx_3 + \frac{a_{14}}{a_{21}}\left(\frac{x_4}{x_1}\right)^{e-1}dx_4.\end{aligned} \tag{7.7}$$

Recall that our section $\sigma^\epsilon \in H^0(S, S^2\widetilde{\Omega}_S(-m))$ can be written in U_0 as

$$\begin{aligned}\sigma_0^\epsilon &= \frac{X_{1234}^\epsilon}{X_0^{e-1-\epsilon_0}}(a_{2134}X_3X_4dX_1dX_2 + a_{2413}X_4X_2dX_1dX_3 + a_{4123}X_2X_3dX_1dX_4 \\ &\quad + a_{2134}X_1X_2dX_3dX_4 + a_{2413}X_3X_1dX_2dX_4 + a_{4123}X_1X_4dX_2dX_3).\end{aligned}$$

Now this corresponds to a section $\sigma^\epsilon \in H^0(S, S^2\Omega_S(-m))$, which can be written

$$\begin{aligned}\sigma_0^\epsilon &= x_{1234}^\epsilon(a_{2134}x_3x_4dx_1dx_2 + a_{2413}x_4x_2dx_1dx_3 + a_{4123}x_2x_3dx_1dx_4 \\ &\quad + a_{2134}x_1x_2dx_3dx_4 + a_{2413}x_3x_1dx_2dx_4 + a_{4123}x_1x_4dx_2dx_3).\end{aligned}$$

Note that any such σ_0^ϵ is a multiple, as an element of $S^2\Omega_S(-m)(S_0)$, of the section

$$\begin{aligned}\sigma_0 = & a_{2134}x_3x_4dx_1dx_2 + a_{2413}x_4x_2dx_1dx_3 + a_{4123}x_2x_3dx_1dx_4 \\ & + a_{2134}x_1x_2dx_3dx_4 + a_{2413}x_3x_1dx_2dx_4 + a_{4123}x_1x_4dx_2dx_3.\end{aligned}$$

Therefore, as long as we are looking at the base locus, only the section σ_0 will be relevant to us. We will now focus on σ_0. Using formulas (7.6) and (7.7), to substitute the x_1, x_2, dx_1, dx_2, it becomes a straightforward (but tedious) computation to obtain

$$\sigma_0^\epsilon = \frac{1}{a_{21}(x_1x_2)^{e-1}}\left(a_{3123}^{40}x_4x_3^{e-1}dx_3^2 + a_{2414}^{30}x_3x_4^{e-1}dx_4^2 + (a_{1002}^{34} + a_{2331}^{04}x_3^e + a_{2414}^{03}x_4^e)dx_3dx_4\right),$$

where $a_{pqrs}^{tu} := a_{pq}a_{rs}a_{tu}$. With this, one can compute the base locus. If we look at the constructed sections as a sections in $H^0(S, S^2\Omega_S)$, then the base locus of all this family is a finite set of points in S. If on looks at those sections as sections in $H^0(\mathbb{P}(\Omega_S), \mathcal{O}_{\mathbb{P}(\Omega_S)}(2))$, then the base locus is a divisor in $\mathbb{P}(\Omega_S)$. The fact that in $\mathbb{P}(\Omega_S)$ the base locus is a divisor is unfortunate. Indeed, if it was for example a finite number of points, then we would have deduced that Ω_S was nef. The hope of such a result was in fact the motivation for all this chapter.

7.5.3 A small deformation of a Fermat-type complete intersection in $\mathbb{P}^4$

We will, in this section, study a small deformation of the Fermat-type complete intersection. It turns out that one cannot deform the global sections that we constructed in the previous section. Let $e_1 := \lfloor \frac{e}{2} \rfloor$ and $e_2 := \lceil \frac{e}{2} \rceil$, so that $e = e_1 + e_2$. For $\alpha := (\alpha_1, \alpha_2) \in \mathbb{C}^2$ and $\beta := (\beta_1, \beta_2) \in \mathbb{C}^2$, we set

$$\begin{aligned} F_\alpha &:= F + \alpha_1 X_0^{e_1}X_1^{e_2} + \alpha_2 X_2^{e_1}X_3^{e_2}, \\ G_\beta &:= G + \beta_1 X_0^{e_1}X_1^{e_2} + \beta_2 X_2^{e_1}X_3^{e_2}. \end{aligned}$$

Define $S_\beta^\alpha := (F_\alpha = 0) \cap (G_\beta = 0)$, such that $S_0^0 = S$. It is now an easy application of Theorem 7.4.2 to see that $H^0(S_\beta^\alpha, S^2\widetilde{\Omega}_{S_\beta^\alpha}) = 0$. Recall that that theorem states that

$$H^0(S_\beta^\alpha, S^2\widetilde{\Omega}_{S_\beta^\alpha}) = K_{\cdot dF_\alpha}(0, 4e) \cap K_{\cdot dG_\beta}(0, 4e).$$

We will now prove the vanishing of this group simply by computing $K_{\cdot dF_\alpha}(0, 4e) \cap K_{\cdot dG_\beta}(0, 4e)$. First we compute the partial derivatives:

$$\begin{aligned} \frac{\partial F_\alpha}{\partial X_0} &= eX_0^{e-1} + e_1\alpha_1 X_0^{e_1-1}X_1^{e_2}, \quad \frac{\partial F_\alpha}{\partial X_2} = eX_2^{e-1} + e_1\alpha_2 X_2^{e_1-1}X_3^{e_2}, \\ \frac{\partial F_\alpha}{\partial X_1} &= eX_1^{e-1} + e_2\alpha_1 X_0^{e_1}X_1^{e_2-1}, \quad \frac{\partial F_\alpha}{\partial X_3} = eX_3^{e-1} + e_2\alpha_2 X_2^{e_1}X_3^{e_2-1}, \\ \frac{\partial F_\alpha}{\partial X_4} &= eX_4^{e-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial G_\beta}{\partial X_0} &= ea_0X_0^{e-1} + e_1\beta_1 X_0^{e_1-1}X_1^{e_2}, \quad \frac{\partial G_\beta}{\partial X_2} = ea_2X_2^{e-1} + e_1\beta_2 X_2^{e_1-1}X_3^{e_2}, \\ \frac{\partial G_\beta}{\partial X_1} &= ea_1X_1^{e-1} + e_2\beta_1 X_0^{e_1}X_1^{e_2-1}, \quad \frac{\partial G_\beta}{\partial X_3} = ea_3X_3^{e-1} + e_2\beta_2 X_2^{e_1}X_3^{e_2-1}, \\ \frac{\partial G_\beta}{\partial X_4} &= ea_4X_4^{e-1}. \end{aligned}$$

Observe now that

$$K_{\cdot dF_\alpha}(0,4e) = \bigcap_{i=0}^{4} K_{\cdot \frac{\partial F_\alpha}{\partial X_i}}(0,4e) \quad \text{and} \quad K_{\cdot dG_\beta}(0,4e) = \bigcap_{i=0}^{4} K_{\cdot \frac{\partial G_\beta}{\partial X_i}}(0,4e).$$

Now take an element

$$\xi := \sum_{\substack{I \\ |I|=4e-5}} \xi^I \frac{1}{X^{I+\mathbb{I}}} \in K_{\cdot dF_\alpha}(0,4e) \cap K_{\cdot dG_\beta}(0,4e) \subseteq H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-4e)).$$

Now we simply make a Gauss algorithm type argument:

$$\xi \in K_{\cdot \frac{\partial F_\alpha}{\partial X_0}} \cap K_{\cdot \frac{\partial G_\beta}{\partial X_0}} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot \frac{\partial F_\alpha}{\partial X_0} & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot \frac{\partial G_\beta}{\partial X_0} & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (eX_0^{e-1} + e_1\alpha_1 X_0^{e_1-1}X_1^{e_2}) & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (ea_0X_0^{e-1} + e_1\beta_1 X_0^{e_1-1}X_1^{e_2}) & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_0^{e-1} & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_0^{e_1-1}X_1^{e_2} & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{ll} \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_0 \geqslant e-1 \\ \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_0 \geqslant e_1 - 1 \text{ and } \mathrm{i}_1 \geqslant \mathrm{e}_2 \end{array} \right.$$

By the same computation, we get

$$\xi \in K_{\cdot \frac{\partial F_\alpha}{\partial X_1}} \cap K_{\cdot \frac{\partial G_\beta}{\partial X_1}} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (eX_1^{e-1} + e_2\alpha_1 X_0^{e_1}X_1^{e_2-1}) & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (ea_1X_1^{e-1} + e_2\beta_1 X_0^{e_1}X_1^{e_2-1}) & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_1^{e-1} & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_0^{e_1}X_1^{e_2-1} & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{ll} \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_1 \geqslant e-1 \\ \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_0 \geqslant e_1 \text{ and } \mathrm{i}_1 \geqslant \mathrm{e}_2 - 1 \end{array} \right.$$

$$\xi \in K_{\cdot \frac{\partial F_\alpha}{\partial X_2}} \cap K_{\cdot \frac{\partial G_\beta}{\partial X_2}} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (eX_2^{e-1} + e_1\alpha_1 X_2^{e_1-1}X_3^{e_2}) & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot (ea_2X_2^{e-1} + e_1\beta_1 X_2^{e_1-1}X_3^{e_2}) & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_2^{e-1} & = & 0 \\ \sum_I \xi^I \frac{1}{X^{I+\mathbb{I}}} \cdot X_2^{e_1-1}X_3^{e_2} & = & 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{ll} \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_2 \geqslant e-1 \\ \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_2 \geqslant e_1 - 1 \text{ and } \mathrm{i}_3 \geqslant \mathrm{e}_2 \end{array} \right.$$

$$\xi \in K_{\cdot \frac{\partial F_\alpha}{\partial X_3}} \cap K_{\cdot \frac{\partial G_\beta}{\partial X_3}} \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_3 \geqslant e-1 \\ \xi^I = 0 & \forall I = (i_0, \cdots, i_4) \ / \ i_2 \geqslant e_1 \text{ and } \mathrm{i}_3 \geqslant \mathrm{e}_2 - 1 \end{array} \right.$$

$$\xi \in K_{\cdot \frac{\partial F_\alpha}{\partial X_4}} \cap K_{\cdot \frac{\partial G_\beta}{\partial X_4}} \quad \Leftrightarrow \quad \left\{ \ \xi^I = 0 \qquad \forall I = (i_0, \cdots, i_4) \ / \ i_4 \geqslant e-1 \right.$$

Now to conclude, it suffices to see that every multi-index I such that $|I| = 4e + a - 5$ has to satisfy at least one of the above conditions. To see this, suppose that there is an $I = (i_0, \cdots, i_4)$ satisfying

$$\begin{array}{lllll} & i_\ell \leqslant e-2 \quad \forall 0 \leqslant \ell \leqslant 4 & & & \\ \text{and} & (i_0 \leqslant e_1 - 2 \ \ \mathit{or} \ \ i_1 \leqslant e_2 - 1) & \text{and} & (\mathrm{i}_0 \leqslant \mathrm{e}_1 - 1 \ \ \mathit{or} \ \ \mathrm{i}_1 \leqslant \mathrm{e}_2 - 2) \\ \text{and} & (i_2 \leqslant e_1 - 2 \ \ \mathit{or} \ \ i_3 \leqslant e_2 - 1) & \text{and} & (\mathrm{i}_2 \leqslant \mathrm{e}_1 - 1 \ \ \mathit{or} \ \ \mathrm{i}_3 \leqslant \mathrm{e}_2 - 2) \end{array}$$

This would imply that

$$4e - 5 = i_0 + \cdots + i_4 \leqslant e_2 - 2 + e_2 - 2 + e - 2 + e - 2 + e - 2 = 4e - 9.$$

This proves that the kernel is trivial, and therefore, $H^0(S^\alpha_\beta, S^2\Omega_{S^\alpha_\beta}) = 0$.

Generic vanishing

We would like to mention that using a simple semicontinuity argument we get

Corollary 7.5.3. *Let $e \geqslant 5$. A generic complete intersection surface $X \subseteq \mathbb{P}^4$ of type (e, e) satisfies*

$$H^0(X, S^2\widetilde{\Omega}_X) = H^0(X, S^2\Omega_X) = 0.$$

There are many such generic vanishing results that can be obtained, by taking some very particular examples and applying Theorem 7.4.2 or 7.4.1.

Chapter 8

Differential equations as embedding obstructions and vanishing theorems

Recall that in [Div08], Diverio established a vanishing theorem for Green-Griffiths jet differentials on complete intersection varieties.

Theorem 8.0.4 (Diverio)**.** *Let $X \subseteq \mathbb{P}^N$ be a smooth complete intersection. Then*

$$H^0(X, E^{GG}_{k,m}\Omega_X) = 0$$

for all $m \geqslant 1$ and $1 \leqslant k < \dim X/\operatorname{codim} X$.

His proof is based on a vanishing theorem due to Bruckman and Rackwitz for Schur powers of the cotangent bundle of a complete intersection variety in [BR90].

Moreover, Schneider [Sch92] proved some vanishing theorems concerning the symmetric powers of the cotangent bundle of a projective variety (not necessarily a complete intersection).

Theorem 8.0.5 (Schneider [Sch92] Theorems 1.1 and 1.2)**.** *Let $X \subset \mathbb{P}^N$ be an n-dimensional smooth variety. Let $m \in \mathbb{Z}$.*

1. *If $q \leqslant 2n - N - 1$ and $k \geqslant m + 2$, then*
$$H^q(X, S^k\Omega_X \otimes \mathcal{O}_X(m)) = 0.$$
2. *If $2n > N$ and $k \geqslant m + 1$, then*
$$H^0(X, S^k\Omega_X \otimes \mathcal{O}_X(m)) = 0.$$

The aim of this chapter is to present generalized versions of Schneider's result. This will also allow us to generalize Diverio's result and unify Schneider's and Diverio's theorems (see Theorem 8.2.3 and Corollary 8.3.2). With the same argument, we can also generalize a vanishing theorem due to Pacienza-Rousseau concerning their generalized Green-Griffiths jet differential bundle (see Corollary 8.4.2). This proof will also illustrate how the bundle $\widetilde{\Omega}$ is useful (at least formally) in such circumstances.

8.1 A cohomological lemma

During the proof, we will need an elementary cohomological lemma.

Lemma 8.1.1. *Let X be a projective variety. Let G be a vector bundle on X. Let $c \geqslant 1$. Suppose we have k long exact sequences of vector bundles on X:*

$$\begin{array}{c} 0 \to E_1^\ell \to E_1^{\ell-1} \to \quad \cdots \quad \to E_1^1 \to E_1^0 \to F_1 \to 0 \\ 0 \to E_2^\ell \to E_2^{\ell-1} \to \quad \cdots \quad \to E_2^1 \to E_2^0 \to F_2 \to 0 \\ \vdots \\ 0 \to E_k^\ell \to E_k^{\ell-1} \to \quad \cdots \quad \to E_k^1 \to E_k^0 \to F_k \to 0. \end{array}$$

Fix an integer q. Suppose that

$$H^j(X, E_1^{i_1} \otimes \cdots \otimes E_k^{i_k} \otimes G) = 0 \quad \text{for all} \quad j \leqslant q + i_1 + \cdots + i_k.$$

Then

$$H^j(X, F_1 \otimes \cdots \otimes F_k \otimes G) = 0 \quad \text{for all} \quad j \leqslant q.$$

Remark 8.1.2. The case $k = 1$ appears already in Schneider's article [Sch92].

Proof. The proof is straightforward, but we write it down for the sake of completeness. This is just an induction on k.

Let us start when $k = 1$. We proceed by induction on ℓ.
When $\ell = 0$, this is obvious. When $\ell = 1$, we just have one short exact sequence

$$0 \to E_1^1 \to E_1^0 \to F_1 \to 0.$$

Tensoring it by G and looking at the associated long exact sequence in cohomology gives the result. Now when $\ell \geqslant 2$, we cut the long exact sequence into two pieces, and tensor everything by G to obtain

$$\begin{array}{l} 0 \to E_1^\ell \otimes G \to E_1^{\ell-1} \otimes G \to \quad K \otimes G \quad \to 0 \\ \qquad\qquad\qquad\qquad\qquad 0 \to \quad K \otimes G \quad \to E_1^{\ell-2} \otimes G \to \cdots \to E_1^1 \otimes G \to E_1^0 \otimes G \to F_1 \otimes G \to 0. \end{array}$$

Apply the global section functor to the first exact sequence to obtain $H^j(X, K \otimes G) = 0$ for all $j \leqslant q + \ell - 1$, and then apply the induction hypothesis.

Now we let $k \geqslant 2$. Suppose the result holds for any family of k exact sequences. Take one more exact sequence

$$0 \to E_{k+1}^\ell \to E_{k+1}^{\ell-1} \to \cdots \to E_{k+1}^1 \to E_{k+1}^0 \to F_{k+1} \to 0. \tag{8.1}$$

Suppose that $H^j(X, E_1^{i_1} \otimes \cdots \otimes E_{k+1}^{i_{k+1}} \otimes G) = 0$ for all $j \leqslant q + i_1 + \cdots + i_{k+1}$. Tensoring the exact sequence (8.1) by $F_1 \otimes \cdots \otimes F_k \otimes G$, we obtain

$$0 \to \widetilde{E}_{k+1}^\ell \to \widetilde{E}_{k+1}^{\ell-1} \to \cdots \to \widetilde{E}_{k+1}^1 \to \widetilde{E}_{k+1}^0 \to F_1 \otimes \cdots \otimes F_{k+1} \otimes G \to 0,$$

where $\widetilde{E}_{k+1}^i := E_{k+1}^i \otimes F_1 \otimes \cdots \otimes F_k \otimes G$. Therefore, to prove that $H^j(X, F_1 \otimes \cdots \otimes F_{k+1} \otimes G) = 0$ for all $j \leqslant q$, it suffices to prove that for any $0 \leqslant i \leqslant \ell$,

$$H^j(X, \widetilde{E}_{k+1}^i) = 0$$

for all $j \leqslant q + i$. To do so, fix $0 \leqslant i \leqslant \ell$. Consider the k long exact sequences

$$\begin{array}{c} 0 \to E_1^\ell \otimes E_{k+1}^i \to E_1^{\ell-1} \otimes E_{k+1}^i \to \quad \cdots \quad \to E_1^1 \otimes E_{k+1}^i \to E_1^0 \otimes E_{k+1}^i \to F_1 \otimes E_{k+1}^i \to 0 \\ 0 \to E_2^\ell \to E_2^{\ell-1} \to \quad \cdots \quad \to E_2^1 \to E_2^0 \to F_2 \to 0 \\ \vdots \\ 0 \to E_k^\ell \to E_k^{\ell-1} \to \quad \cdots \quad \to E_k^1 \to E_k^0 \to F_k \to 0. \end{array}$$

By hypothesis, $H^j(X, E_1^{i_1} \otimes E_{k+1}^{i} \otimes E_2^{i_2} \otimes \cdots \otimes E_k^{i_k} \otimes G) = 0$ for all $j \leqslant q + i_1 + \cdots + i_k + i$. Therefore, by the induction hypothesis, we obtain that $H^j(X, \widetilde{E}_{k+1}^i) = H^j(X, E_{k+1}^i \otimes F_1 \otimes \cdots \otimes F_k \otimes G) = 0$ for all $j \leqslant q + i$. □

Similarly, we obtain the following.

Lemma 8.1.3. *Let X be a projective variety. Let $c \geqslant 1$. Suppose we have k long exact sequences of vector bundles on X:*

$$\begin{array}{c}
0 \to F_1 \to E_1^0 \to E_1^1 \quad \cdots \quad \to E_1^{\ell-1} \to E_1^{\ell} \to 0 \\
0 \to F_2 \to E_2^0 \to E_2^1 \quad \cdots \quad \to E_2^{\ell-1} \to E_2^{\ell} \to 0 \\
\vdots \\
0 \to F_k \to E_k^0 \to E_k^1 \quad \cdots \quad \to E_k^{\ell-1} \to E_k^{\ell} \to 0.
\end{array}$$

Fix an integer q. Suppose that

$$H^j(X, E_1^{i_1} \otimes \cdots \otimes E_k^{i_k} \otimes G) = 0 \quad \textit{for all} \quad j \leqslant q - i_1 - \cdots - i_k.$$

Then

$$H^j(X, F_1 \otimes \cdots \otimes F_k \otimes G) = 0 \quad \textit{for all} \quad j \leqslant q.$$

8.2 Schneider's vanishing revisited

To prove his result, Schneider considered a Kozsul resolution of $S^k\Omega_X$. Then, applying his cohomological lemma and Le Potier's vanishing theorem [LP75], he was able to conclude. Here we follow the same idea in our more general setting. However, we will not be able to apply directly the vanishing theorem of Le Potier: we have to replace it with a more general vanishing theorem. The following theorem appears first in a work of Ein and Lasarsfeld [EL93], but this result follows from Le Potier's theorem thanks to an idea of Manivel. We refer to [EL93] and [Man97] for more details.

Theorem 8.2.1 (Ein-Lazarsfeld/Manivel)**.** *Let X be a smooth projective variety of dimension n. Let $E_1, \ldots, E_r$ be vector bundles on X, of ranks $e_1, \ldots, e_r$, and let A be an ample line bundle on X. Assume that each E_i is generated by global sections, and fix $a_1, \ldots, a_r \geqslant 1$. Then*

$$H^k(X, K_X \otimes \Lambda^{a_i} E_1 \otimes \cdots \otimes \Lambda^{a_r} E_r \otimes A) = 0$$

for $k > (e_1 - a_1) + \cdots + (e_r - a_r)$.

The case $r = 1$ is the original result of Le Potier [LP75].

Before we continue we recall a well-known fact. Consider a subvariety $X \subseteq \mathbb{P}^N$, and let $N_{X/\mathbb{P}^N}$ denote the normal bundle to X in $\mathbb{P}^N$. Then $N_{X/\mathbb{P}^N} \otimes \mathcal{O}_X(-1)$ is globally generated. This follows directly from the normal exact sequence and the Euler exact sequence. As a matter of notation, we fix an $(N+1)$-dimensional complex vector space such that $\mathbb{P}^N = \mathbb{P}(V)$.

Putting all this together, we can prove the following.

Theorem 8.2.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geqslant 1$ and k integers $\ell_1, \ldots, \ell_k \geqslant 0$, and $a \in \mathbb{Z}$. If $\ell_1 + \cdots + \ell_k > a$, then*

$$H^j(X, S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0$$

for $j < n - k \cdot c$.

Proof. Let $N := N_{X/\mathbb{P}^N}$. Recall the tilde conormal exact sequence

$$0 \to N^* \to \widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to \widetilde{\Omega}_X \to 0.$$

Taking different symmetric powers we obtain k exact sequences

$$\begin{array}{c}
0 \to \Lambda^{\ell_1} N^* \to \Lambda^{\ell_1 - 1} N^* \otimes \widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to \quad \cdots \quad \to S^{\ell_1}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to S^{\ell_1}\widetilde{\Omega}_X \to 0 \\
0 \to \Lambda^{\ell_2} N^* \to \Lambda^{\ell_2 - 1} N^* \otimes \widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to \quad \cdots \quad \to S^{\ell_2}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to S^{\ell_2}\widetilde{\Omega}_X \to 0 \\
\vdots \\
0 \to \Lambda^{\ell_k} N^* \to \Lambda^{\ell_k - 1} N^* \otimes \widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to \quad \cdots \quad \to S^{\ell_k}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} \to S^{\ell_k}\widetilde{\Omega}_X \to 0.
\end{array}$$

For $1 \leqslant p \leqslant k$, we let

$$E_p^i \quad = \quad \Lambda^i N^* \otimes S^{\ell_p - i}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} = \Lambda^i N^* \otimes S^{\ell_p - i} V \otimes \mathcal{O}_X(i - \ell_p).$$

By Lemma 8.1.1, to prove that

$$H^j(X, S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0 \ \text{ for } j < n - k \cdot c,$$

it suffices to prove that

$$H^j(X, E_1^{i_1} \otimes \cdots \otimes E_k^{i_k} \otimes \mathcal{O}_X(a)) = 0 \ \text{ for } j < n - k \cdot c + i_1 + \cdots + i_k.$$

We now prove this fact. We have:

$$\begin{aligned}
H^j(\, X, E_1^{i_1} \otimes \cdots \otimes E_k^{i_k} \otimes \mathcal{O}_X(a)) &= H^j(X, \Lambda^{i_1} N^* \otimes S^{\ell_1 - i_1}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} \otimes \cdots \otimes \Lambda^{i_k} N^* \otimes S^{\ell_k - i}\widetilde{\Omega}_{\mathbb{P}^N_{|X}} \otimes \mathcal{O}_X(a)) \\
&= H^j(X, \Lambda^{i_1} N^* \otimes \cdots \otimes \Lambda^{i_k} N^* \otimes S^{\ell_1 - i_1} V \otimes \cdots \otimes S^{\ell_k - i_k} V \otimes \mathcal{O}_X(i_1 - \ell_1 + \cdots + i_k - \ell_k + a)) \\
&= S^{\ell_1 - i_1} V \otimes \cdots \otimes S^{\ell_k - i_k} V \otimes H^j(X, \Lambda^{i_1}(N^*(1)) \otimes \cdots \otimes \Lambda^{i_k}(N^*(1)) \otimes \mathcal{O}_X(a - \ell_1 - \cdots - \ell_k)).
\end{aligned}$$

On the other hand, using Serre duality, we obtain

$$\begin{aligned}
&H^j(\, X, \Lambda^{i_1}(N^*(1)) \otimes \cdots \otimes \Lambda^{i_k}(N^*(1)) \otimes \mathcal{O}_X(a - \ell_1 - \cdots - \ell_k) \\
&\quad = \quad H^{n-j}(X, \Lambda^{i_1}(N(-1)) \otimes \cdots \otimes \Lambda^{i_k}(N(-1)) \otimes \mathcal{O}_X(\ell_1 + \cdots + \ell_k - a) \otimes K_X).
\end{aligned}$$

Since $\mathcal{O}_X(\ell_1 + \cdots + \ell_k - a)$ is ample and $N(-1)$ is a rank-c globally generated vector bundle, we can apply Theorem 8.2.1 to see that this last cohomology group vanishes if

$$n - j > k \cdot c - i_1 - \cdots - i_k$$

or equivalently, if

$$j < n - k \cdot c + i_1 + \cdots + i_k.$$

This is exactly what was needed to conclude the proof. □

We are now in a position to deduce the announced generalization of Schneider's theorem.

Theorem 8.2.3. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geqslant 1$ and k integers $\ell_1, \ldots, \ell_k \geqslant 0$, and $a \in \mathbb{Z}$.*

1. *If $\ell_1 + \cdots + \ell_k > a + k$, then*

$$H^j(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0$$

for $j < n - k \cdot c$.

2. *If* $\ell_1 + \cdots + \ell_k > a$*, then*

$$H^0(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0$$

if $0 < n - k \cdot c$*.*

Remark 8.2.4. The particular case $k = 1$ is precisely Schneider's result.

Proof. Recall that we have the Euler exact sequence

$$0 \to \Omega_X \to \widetilde{\Omega}_X \to \mathcal{O}_X \to 0.$$

Therefore we have an injection

$$0 \to S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a) \to S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a).$$

Assertion 2 then follows from the inclusion

$$H^0(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) \subseteq H^0(X, S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a))$$

and we just apply Theorem 8.2.2.

To see assertion 1, we consider the k short exact sequences

$$\begin{array}{lcl} 0 \to S^{\ell_1}\Omega_X \to & S^{\ell_1}\widetilde{\Omega}_X & \to S^{\ell_1-1}\widetilde{\Omega}_X \to 0 \\ 0 \to S^{\ell_2}\Omega_X \to & S^{\ell_2}\widetilde{\Omega}_X & \to S^{\ell_2-1}\widetilde{\Omega}_X \to 0 \\ & \vdots & \\ 0 \to S^{\ell_k}\Omega_X \to & S^{\ell_k}\widetilde{\Omega}_X & \to S^{\ell_k-1}\widetilde{\Omega}_X \to 0. \end{array}$$

By Lemma 8.1.3, it is now sufficient to check that

$$H^j(X, S^{\alpha_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\alpha_k}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0$$

for $j < n - k \cdot c$, where $\ell_i - 1 \leqslant \alpha_i \leqslant \ell_i$. This follows from Theorem 8.2.2. Observe that the hypothesis $\ell_1 + \cdots + \ell_k > a + k$ is needed to cover the case $\alpha_i = \ell_i - 1$ for all $1 \leqslant i \leqslant k$. □

8.3 Application to Green-Griffiths jet differentials

Now we apply those vanishing results to Green-Griffiths jet differential bundles. The idea of converting a vanishing result for the cohomology of symmetric differential form bundles into a vanishing result for the cohomology of Green-Griffiths jet differential bundles is due to Diverio. We therefore proceed along the lines of [Div08].

We start by recalling a cohomological lemma (see [Div08]).

Lemma 8.3.1. *Let* $E \to X$ *be a holomorphic vector bundle with a filtration* $\{0\} = E_r \subset \cdots \subset E_1 \subset E_0 = E$*. If* $H^q(X, \mathrm{Gr}^\bullet E) = 0$*, then* $H^q(X, E) = 0$*.*

Combining this lemma with Theorem 8.2.3, one derives our result:

Corollary 8.3.2. *Let* $X \subseteq \mathbb{P}^N$ *be a smooth variety of dimension* n *and codimension* c*. Let* $a \in \mathbb{Z}$*.*

1. *If* $j < n - k \cdot c$ *and* $a + k < \frac{m}{k}$*, then*

$$H^j(X, E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

2. If $k < \frac{n}{c}$ and $a < \frac{m}{k}$, then

$$H^0(X, E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Proof. Now recall that the Green-Griffiths bundle admits a filtration whose graded bundle is given by

$$\mathrm{Gr}^\bullet E^{GG}_{k,m}\Omega_X = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}\Omega_X \otimes S^{\ell_2}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X.$$

By twisting everything by $\mathcal{O}_X(a)$, we obtain a filtration for $E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}_X(a)$ whose graded bundle is given by

$$\mathrm{Gr}^\bullet E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}_X(a) = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}\Omega_X \otimes S^{\ell_2}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a).$$

We prove assertion 1. By Lemma 8.3.1, we only have to prove that

$$H^j(X, S^{\ell_1}\Omega_X \otimes S^{\ell_2}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \mathcal{O}_X(a)) = 0$$

for all $j < n - k \cdot c$ and $\ell_1 + \cdots + k\ell_k = m$. The hypothesis then gives us

$$a + k < \frac{m}{k} = \frac{\ell_1 + \cdots + k\ell_k}{k} \leqslant \frac{k\ell_1 + \cdots + k\ell_k}{k} = \ell_1 + \cdots + \ell_k.$$

Thus we can conclude by applying assertion 1 of Theorem 8.2.3. The proof of assertion 2 is done in the same way by applying assertion 2 of Theorem 8.2.3 ☐

Remark 8.3.3. When $k = 1$ this result is exactly Schneider's theorem. When X is a complete intersection and $a = 0$, assertion 2 is precisely Diverio's theorem.

8.4 Application to generalized Green-Griffiths jet differentials

In [PR08], Pacienza and Rousseau generalized Green-Griffiths jet differential bundles (see section 1.4.5) and, among other things, they generalized Diverio's vanishing theorem to those bundles.

Theorem 8.4.1 (Pacienza-Rousseau [PR08] Theorem 5.1)**.** *Let $X \subseteq \mathbb{P}^N$ ba a smooth complete intersection. Then*

$$H^0(X, E^{GG}_{p,k,m}) = 0,$$

for all $m \geqslant 1$ and k such that

$$\binom{k+p}{p} - 1 < \frac{\dim(X)}{\mathrm{codim}(X)}.$$

Their proof is based on the same idea than Diverio's proof of Theorem 8.0.4. It is therefore not surprising that one can prove a more general statement using Theorem 8.2.3.

Corollary 8.4.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension c. Let $a \in \mathbb{Z}$.*

1. If

$$j < N - \binom{k+p}{p} \cdot c \quad \textit{and} \quad a + \binom{k+p}{p} - 1 < \frac{m}{k},$$

then

$$H^j(X, E^{GG}_{p,k,m} \otimes \mathcal{O}_X(a)) = 0,$$

2. If $\binom{k+p}{p} < \frac{N}{c}$ and $a < \frac{m}{k}$, then

$$H^0(X, E^{GG}_{p,k,m}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Proof. Recall ([PR08] Remark 2.6) that there is a filtration for $E^{GG}_{p,k,m}\Omega_X \otimes \mathcal{O}_X(a)$ such that the graded terms are of the form

$$\bigotimes_{\alpha\in I_1} S^{q^1_\alpha}\Omega_X \cdots \bigotimes_{\alpha\in I_k} S^{q^k_\alpha}\Omega_X \otimes \mathcal{O}_X(a), \tag{8.2}$$

where $I_\ell := \{\alpha = (\alpha_1, \ldots, \alpha_p) \;/\; \alpha_1 + \cdots + \alpha_p = \ell\}$ and where

$$\sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha \alpha = (m, \ldots, m).$$

First observe that

$$|I_1| + \cdots + |I_k| = \binom{k+p}{p} - 1.$$

Therefore there are only $\binom{k+p}{p}-1$ symmetric powers in (8.2). In order to prove assertion 1, we apply assertion 1 of Theorem 8.2.3, and therefore, with the above observation, we now just have to prove that under our hypothesis,

$$\sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha > a + \binom{k+p}{p} - 1.$$

But this follows at once from

$$\begin{aligned}\frac{(m, \ldots, m)}{k} &= \sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha \frac{\alpha}{k} \\ &\leq \sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha \frac{(k, \ldots, k)}{k} = \left(\sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha, \ldots, \sum_{\ell=1}^{k} \sum_{\alpha\in I_\ell} q^\ell_\alpha\right)\end{aligned}$$

and similarly for the proof of assertion 2. □

8.5 A further generalization

One can generalize Theorem 8.2.2 and 8.2.3 even further if one uses the whole strength of Manivel's results.

Theorem 8.5.1 (Manivel [Man97] Theorem A)**.** *Let E be a holomorphic vector bundle of rank e and let L be a line bundle on a smooth projective complex variety X of dimension n. Suppose that E is ample and L is nef, or that E is nef and L ample. Then, for any sequences of integers $k_1, \ldots, k_\ell$ and $j_1, \ldots, j_m$,*

$$H^{p,q}(X, S^{k_1}E \otimes \cdots \otimes S^{k_\ell}E \otimes \Lambda^{j_1}E \otimes \cdots \otimes \Lambda^{j_m}E \otimes (\det E)^{\ell+n-p} \otimes L) = 0$$

as soon as $p + q > n + \sum_{s=1}^{m}(e - j_s)$.

Redoing the proof of Theorem 8.2.2 and 8.2.3, and considering moreover exterior powers of $\widetilde{\Omega}_X$ and Ω_X, one obtains the following.

Theorem 8.5.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider integers $k, r \geqslant 0$, integers $\ell_1, \ldots, \ell_k, m_1, \ldots, m_r \geqslant 0$, and $a \in \mathbb{Z}$.*

1. *If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > a + (n+1)(r+p)$ and $j + p < n - kc$, then*

$$H^{p,j}(X, S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \Lambda^{m_1}\widetilde{\Omega}_X \otimes \cdots \otimes \Lambda^{m_r}\widetilde{\Omega}_X \otimes K_X^{-(r+p)} \otimes \mathcal{O}_X(a)) = 0.$$

2. If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > k + a + (n+1)(r+p)$ and $j + p < n - kc$, then

$$H^{p,j}(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \Lambda^{m_1}\Omega_X \otimes \cdots \otimes \Lambda^{m_r}\Omega_X \otimes K_X^{-(r+p)} \otimes \mathcal{O}_X(a)) = 0.$$

3. If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > a + (n+1)(r+p)$ and $p < n - kc$, then

$$H^{p,0}(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \Lambda^{m_1}\Omega_X \otimes \cdots \otimes \Lambda^{m_r}\Omega_X \otimes K_X^{-(r+p)} \otimes \mathcal{O}_X(a)) = 0.$$

If one makes some more assumptions on K_X, one can obtain a better statement. For example, if one suppose X to be Fano, one can obtain a result without the powers of the canonical bundle.

Theorem 8.5.3. *Let $X \subseteq \mathbb{P}^N$ be a Fano variety of dimension n and codimension $c = N - n$. Consider integers $k, r \geqslant 0$, integers $\ell_1, \dots, \ell_k, m_1, \dots, m_r \geqslant 0$, and $a \in \mathbb{Z}$.*

1. If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > a + (n+1)(r+p)$ and $j + p < n - kc$, then

$$H^{p,j}(X, S^{\ell_1}\widetilde{\Omega}_X \otimes \cdots \otimes S^{\ell_k}\widetilde{\Omega}_X \otimes \Lambda^{m_1}\widetilde{\Omega}_X \otimes \cdots \otimes \Lambda^{m_r}\widetilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0.$$

2. If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > k + a + (n+1)(r+p)$ and $j + p < n - kc$, then

$$H^{p,j}(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \Lambda^{m_1}\Omega_X \otimes \cdots \otimes \Lambda^{m_r}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

3. If $\ell_1 + \cdots + \ell_k + m_1 + \cdots + m_r > a + (n+1)(r+p)$ and $p < n - kc$, then

$$H^{p,0}(X, S^{\ell_1}\Omega_X \otimes \cdots \otimes S^{\ell_k}\Omega_X \otimes \Lambda^{m_1}\Omega_X \otimes \cdots \otimes \Lambda^{m_r}\Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Chapter 9

Cohomology of line bundles

One of our main approaches to understand the positivity of the cotangent bundle of a complete intersection variety was to compute its cohomology. This leads us in a natural way to try to compute the cohomology of the symmetric powers of the cotangent bundle (and relative cotangent bundle) of the universal complete intersection. The first step in doing this is just to understand the cohomology of line bundles on a universal hypersurface. This problem seems already interesting in itself. We will here give some results concerning those cohomology groups. Let us first fix our notation. We let V be an $(N+1)$-dimensional complex vector space. We then set $\mathbb{P}^N := \mathbb{P}(V)$ and $\mathbb{P}^{N_d} := \mathbb{P}(S^d V^*) = \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^*)$. Let then $\mathbf{P} := \mathbb{P}^{N_d} \times \mathbb{P}^N$ and

$$\mathcal{H} := \{(t,x) \in \mathbf{P} \ / \ t(x) = 0\}$$

The problem we approach in this section is the determination of the groups $H^i(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2))$ when $m_1, m_2 \in \mathbb{Z}$.

9.1 Motivation

We start by explaining more precisely how this problem arose while we were working on Debarre's conjecture. As we said, given a complete intersection variety $X \subseteq \mathbb{P}^N$ our strategy was to understand the cohomology group $H^i(X, S^m\Omega_X \otimes \mathcal{O}_X(a))$. Our gaol was to prove that, under the hypothesis of Debarre's conjecture, these groups vanish if $a \in \mathbb{Z}$ and $m \gg 0$, which would prove the conjecture. One idea was to use representation theory.

First let us illustrate how one can use representation theory to compute cohomology groups. Suppose we want to understand the groups $H^i(\mathbb{P}^N, T\mathbb{P}^N)$ for $m \in \mathbb{N}$. Let $G = \mathrm{Gl}_{N+1}(\mathbb{C})$. We can consider the G-action on $\mathbb{P}^N$. This action extends into an action on $T\mathbb{P}^N$. In particular the cohomology groups $H^i(\mathbb{P}^N, T\mathbb{P}^N)$ are representations of G. And with a long exact sequence argument one can determine exactly what these groups are. Indeed, consider the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to V^* \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to T\mathbb{P}^N \to 0,$$

where V is an $(N+1)$-dimensional vector space such that $\mathbb{P}^N = \mathbb{P}(V)$. Observe that G acts on this exact sequence. More precisely, the vector bundles all have a G-action, and there are compatible G-actions on the morphisms. Then look at the associated long exact sequence in cohomology, which in our situation will have only 3 non-zero terms:

$$0 \to \mathbb{C} \to V^* \otimes V \to H^0(\mathbb{P}^N, T\mathbb{P}^N) \to 0.$$

The important point is that the morphisms are G-equivariant. Now we apply Pieri's formula to see that

$$V^* \otimes V = \det V^* \otimes \Lambda^{N-1} V \otimes V = \det V^* \otimes \left(\Gamma^{(2,1,\dots,1,0)} V \oplus \Lambda^N V\right) = \mathbb{C} \oplus \left(\det V^* \otimes \Gamma^{(2,1,\dots,1,0)} V\right),$$

where $\Gamma^{(2,1,\ldots,1,0)}V$ denotes the irreducible representation of $G = \mathrm{Gl}(V)$ associated to $(2,1,\ldots,1,0)$. And since the map $\mathbb{C} \to V^* \otimes V$ is G-equivariant, we conclude that $H^0(\mathbb{P}^N, T\mathbb{P}^N) \simeq \det V^* \otimes \Gamma^{(2,1,\ldots,1,0)}V$ using Schur's lemma.

On the other hand, one can consider the cohomological side of Debarre's conjecture by a step by step approach. Let $X = H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N$ be a complete intersection variety, where $H_i \in |\mathcal{O}_{\mathbb{P}^N}(d_i)|$ for some $d_i \in \mathbb{N}$. For $1 \leqslant i \leqslant c$ let $X_i = H_1 \cap \cdots \cap H_i$ and let $X_0 = \mathbb{P}^N$. For each $1 \leqslant i \leqslant c$ one can consider the exact sequences

$$0 \to \mathcal{O}_{X_{i-1}}(-d_i) \to \mathcal{O}_{X_{i-1}} \to \mathcal{O}_{X_i} \to 0,$$
$$0 \to \mathcal{O}_{X_i}(-d_i) \to \Omega_{X_{i-1}|X_i} \to \Omega_{X_i} \to 0.$$

By taking suitable symmetric powers of those sequences, by twisting them by suitable line bundles, and by looking at the long exact sequence in cohomology (just as what we did in Chapter 7) one should have all the information we need to understand the groups $H^i(X, S^m\Omega_X \otimes \mathcal{O}_X(a))$. However, as we saw in Chapter 7, this is quiet difficult. The idea is to try the step by step approach using representation theory. Unfortunately, there in no G-action on a compete intersection variety so we have to change our point of view by looking at the universal complete intersection variety. Let $\mathcal{X}$ be the universal complete intersection of multidegree $(d_1, \ldots, d_c)$, and for all $1 \leqslant i \leqslant c$, set

$$\mathcal{X}_i := \{(t_1, \ldots, t_c, x) \in \mathbf{P} \times \mathbb{P}^N \mid \forall\, 1 \leqslant j \leqslant i \quad t_j(x) = 0\},$$

where $\mathbf{P} := \mathbb{P}^{N_{d_1}} \times \cdots \times \mathbb{P}^{N_{d_c}}$. Observes that $\mathcal{X} = \mathcal{X}_c$. Let $p : \mathbf{P} \times \mathbb{P}^N \to \mathbf{P}$ be the projection onto $\mathbf{P}$, for $1 \leqslant i \leqslant c$ denote the projection onto the i-th factor by $p_i : \mathbf{P} \times \mathbb{P}^N \to \mathbb{P}^{N_{d_i}}$ and denote the projection onto the last factor by $q : \mathbf{P} \times \mathbb{P}^N \to \mathbb{P}^N$. There is a G-action on $\mathbf{P} \times \mathbb{P}^N$ given by

$$g \cdot (t_1, \ldots, t_c, x) := (t_1 \circ g, \ldots, t_c \circ g, g^{-1}(x)).$$

Obviously, all the $\mathcal{X}_i$ are stable under this action. We are interested in the relative cotangent bundle $\Omega_{\mathcal{X}/\mathbf{P}}$ since $\Omega_{\mathcal{X}/\mathbf{P}|X_t} \simeq \Omega_{X_t}$. And in particular we are interested in cohomology groups $H^i(\mathcal{X}, S^m\Omega_{\mathcal{X}/\mathbf{P}} \otimes L)$ (and the direct image sheaves $R^ip_*S^m\Omega_{\mathcal{X}/\mathbf{P}} \otimes L$), where L is a line bundle on $\mathcal{X}$. Again, for each $1 \leqslant i \leqslant c$ one can consider the following short exact sequences:

$$0 \to p_i^*\mathcal{O}_{\mathbb{P}^N_{d_i}}(-1) \otimes q^*\mathcal{O}_{\mathbb{P}^N}(-d_i) \to \mathcal{O}_{\mathcal{X}_{i-1}} \to \mathcal{O}_{\mathcal{X}_i} \to 0,$$

$$0 \to p_i^*\mathcal{O}_{\mathbb{P}^N_{d_i}}(-1) \otimes q^*\mathcal{O}_{\mathbb{P}^N}(-d_i) \to \Omega_{\mathcal{X}_{i-1}/\mathbf{P}|\mathcal{X}_i} \to \Omega_{\mathcal{X}_i/\mathbf{P}} \to 0.$$

Observe that there is a natural G-action on those exact sequences. By applying suitable symmetric powers to those exact sequence and by twisting by suitable line bundles and looking at the associated long exact sequences in cohomology, one should have enough information to understand the groups $H^i(\mathcal{X}, S^m\Omega_{\mathcal{X}/\mathbf{P}} \otimes L)$. The hope being that from the use of the G-action and the theory of representation of G one could deduce helpful information, just as what we did above. With this approach, the first step is to compute the cohomology of the line bundles on the universal complete intersection variety, and in particular, the cohomology of the line bundles on the universal hypersurface. This is the starting point of this chapter.

9.2 Cohomology on P

First let us recall a well-known result on the cohomology of line bundles on projective spaces.

Theorem 9.2.1. *With the above notation, we have:*

- *If $m \geqslant 0$, then $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) = S^mV$ and $H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) = 0$ for all $i > 0$.*
- *If $-N \leqslant -m < 0$, then $H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m)) = 0$ for all i.*

- *If $-m \leqslant -N-1$, then $H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m)) = S^{m-N-1}V^*$ and $H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m)) = 0$ for all $i < N$*

Applying Künneth formula to this result, we can readily deduce the dimensions of the $H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2))$.

Corollary 9.2.2. *Let $m_1, m_2 \in \mathbb{Z}$. The only cases for which $H^i(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2))$ are non-zero are the following:*

- *If $m_1 \geqslant 0$ and $m_2 \geqslant 0$, then*
$$H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2)) = S^{m_1}S^dV^* \otimes S^{m_2}V.$$
- *If $m_1 \leqslant -N_d - 1$ and $m_2 \geqslant 0$, then*
$$H^{N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2)) = S^{-m_1-N_d-1}S^dV \otimes S^{m_2}V.$$
- *If $m_1 \geqslant 0$ and $m_2 \leqslant -N-1$, then*
$$H^N(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2)) = S^{m_1}S^dV^* \otimes S^{-m_2-N-1}V^*.$$
- *If $m_1 \leqslant -N_d - 1$ and $m_2 \leqslant -N-1$, then*
$$H^{N+N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2)) = S^{-m_1-N_d-1}S^dV \otimes S^{-m_2-N-1}V^*.$$

9.3 A first look at the cohomology on $\mathcal{H}$

From now on we fix a basis $(e_0, \cdots, e_N)$ for V. We will denote the dual basis on V^* by $(e_0^*, \cdots, e_0^*)$. Also, if $k \in \mathbb{N}$, this induces a basis on S^kV that we will denote by $(e_I)_{|I|=k}$, where as usual, for $I = (i_0, \cdots, i_N)$, we have $e_I = e_0^{i_0} \cdots e_N^{i_N}$. Similarly for the basis $(e_I^*)_{|I|=k}$ for S^kV^*. First we remark that the universal hypersurface is given as the zero locus of a section $\sigma_{\mathcal{H}} \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1, d))$. Under the identification $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1, d)) = S^dV^* \otimes S^dV$, we have :

$$\sigma_{\mathcal{H}} = \sum_{|I|=d} e_I^* \otimes e_I.$$

As $\mathcal{H} = (\sigma_{\mathcal{H}} = 0)$ we obtain the restriction exact sequence,

$$0 \to \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d) \xrightarrow{\cdot \sigma_{\mathcal{H}}} \mathcal{O}_{\mathbf{P}}(m_1, m_2) \to \mathcal{O}_{\mathcal{H}}(m_1, m_2) \to 0.$$

A quick look at the long exact sequence associated with this short exact sequence already gives some information. When $N \geqslant 2$ and $d \geqslant 2$, from the above vanishing we can deduce four exact sequences from the long exact sequence.

$$0 \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0$$

$$\begin{aligned} 0 \to H^{N-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) &\to H^N(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \\ &\xrightarrow{a} H^N(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^N(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0 \end{aligned} \tag{9.1}$$

$$\begin{aligned} 0 \to H^{N_d-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) &\to H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \\ &\xrightarrow{b} H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0 \end{aligned} \tag{9.2}$$

$$0 \to H^{N_d+N-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^{N_d+N}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to H^{N_d+N}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to 0$$

Therefore there are only two cases for which the value of the cohomology groups is not clear:

1. When $m_1 \geqslant 1$ and $m_2 \leqslant -N-1$.

2. When $m_1 \leqslant -N-1$ and $m_2 \geqslant d$.

With this, we can readily check the following.

Proposition 9.3.1. *Let $m_1, m_2 \in \mathbb{Z}$. If (m_1, m_2) does not belong to one of the above two cases then we have the expected isomorphisms. more precisely :*

1. $H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong \frac{H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2))}{\cdot\sigma_{\mathcal{H}}(H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1-1, m_2-d))}$.

2. $H^{N-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong H^N(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1-1, m_2-d))$.

3. $H^N(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong H^N(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2))$.

4. $H^{N_d-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong H^{N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1-1, m_2-2))$.

5. $H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong H^{N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2))$.

6. $H^{N_d+N-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \cong \ker\left(H^{N+N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1-1, m_2-d)) \overset{\cdot\sigma_{\mathcal{H}}}{\to} H^{N+N_d}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(m_1, m_2))\right)$.

7. $H^i(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) = 0$ *for all* $i \neq 0, N-1, N, N_d-1, N_d, N_d+N-1$ *and for all* $m_1, m_2 \in \mathbb{Z}$.

For the two remaining cases it is not clear what we can say. However we would like to raise at this point a question, which we believe to admit a positive answer.

Question. Are all the morphisms a and b in the long exact sequences (9.1) and (9.2) associated to the restriction short exact sequence of maximal rank?

This would lead to a full understanding of the cohomology groups of line bundles on the universal hypersurface. We were not able able to answer in general to this question, but we will treat the special case $N = 1$ in the upcoming sections.

9.4 Applications between cohomology groups

Here we would like to start a study of the question we raised. It should be noted that this situation comes with a $G := \mathrm{Gl}_{N+1}(\mathbb{C})$ action on it. In particular all those cohomology groups are in fact representations of G and the morphisms between them are G-equivariant. It is therefore tempting to use representation theory to simplify this problem. Unfortunately we were unable to do so, mainly because we do not know the decomposition of $S^k S^d V$ into irreducible representations: this problem is known as “Plethysm” and it still seems a widely open problem in representation theory and Schur polynomials. We will take here a more naive approach using only basic linear algebra. Therefore we have to start by writing down explicitly what those applications are. Take homogenous coordinates $[X_0 : \cdots : X_N]$ on $\mathbb{P}^N$ coming from the basis $(e_i)_{0 \leqslant i \leqslant N}$. If we look at the Cech cohomological proof of Serre's Theorem (see [Har77]) we obtain the isomorphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) = S^m V = \bigoplus_{|I|=m} \mathbb{C} \cdot X^I$$

for any $m \geqslant 0$, where as usual $I = (i_0, \ldots, i_N)$ are multi-indices. If we now take $-m \leqslant -N-1$, we obtain an isomorphism

$$H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-m)) = S^{m-N-1} V^* = \bigoplus_{|I|=m-N-1} \mathbb{C} \cdot \frac{1}{X^{I+\mathbb{I}}},$$

where $\mathbb{I} := (1, \cdots, 1)$ is the multi-index containing only 1's. Similarly on $\mathbb{P}^{N_d}$ take homogeneous coordinates $[\alpha_I]_{|I|=d}$ coming from the basis $(e_I)_{|I|=d}$. Take a total order $\dashv$ on the set of multi-indices (for example the lexicographic order). We then have isomorphisms

$$H^0(\mathbb{P}^{N_d}, \mathcal{O}_{\mathbb{P}^{N_d}}(m)) \simeq S^m S^d V = \bigoplus_{\substack{I_1 \dashv \cdots \dashv I_m \\ |I_i|=d}} \mathbb{C} \cdot \alpha_{I_1} \cdots \alpha_{I_m}$$

if $m \geqslant 0$, and

$$H^{N_d}(\mathbb{P}^{N_d}, \mathcal{O}_{\mathbb{P}^{N_d}}(m)) = S^{-m-N_d-1} S^d V^* = \bigoplus_{\substack{I_1 \dashv \cdots \dashv I_{-m-N_d-1} \\ |I_i|=d}} \mathbb{C} \cdot \frac{1}{\alpha_{I_1} \cdots \alpha_{I_{-m-N_d-1}} \underline{\alpha}}$$

if $m \leqslant -N_d - 1$, where $\underline{\alpha} := \prod_{|I|=d} \alpha_I$ is the product of all the variables. With this in mind we turn now to the study of the two cases not covered by proposition 9.3.1. We look at the case $m_1 \geqslant 1$ and $m_2 \leqslant -N-1$.

The equation defining $\mathcal{H}$ is exactly

$$F_{\mathcal{H}} = \sum_{|K|=d} \alpha_K X^K.$$

This corresponds to $\sigma_{\mathcal{H}}$ under the above isomorphisms.

Moreover using Kunneth's formula we obtain,

$$\begin{aligned} H^N(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) &= H^0(\mathbb{P}^{N_d}, \mathcal{O}_{\mathbb{P}^{N_d}}(m_1 - 1)) \otimes H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m_2 - d)) \\ &= \bigoplus_{\substack{I_1 \dashv \cdots \dashv I_{m_1-1} \\ |I_i|=d \ \forall i \\ |J|=m_2-d-N-1}} \mathbb{C} \cdot \frac{\alpha_{I_1} \cdots \alpha_{I_{m_1-1}}}{X^{J+\mathbb{I}}}. \end{aligned}$$

And similarly,

$$H^N(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) = \bigoplus_{\substack{I_1 \dashv \cdots \dashv I_{m_1} \\ |I_i|=d \ \forall i \\ |J|=m_2-N-1}} \mathbb{C} \cdot \frac{\alpha_{I_1} \cdots \alpha_{I_{m_1}}}{X^{J+\mathbb{I}}}.$$

The morphism a appearing in the long exact sequence is then exactly

$$\begin{array}{ccc} \displaystyle\bigoplus_{\substack{I_1 \dashv \cdots \dashv I_{m_1-1} \\ |I_i|=d \ \forall i \\ |J|=m_2-d-N-1}} \mathbb{C} \cdot \frac{\alpha_{I_1} \cdots \alpha_{I_{m_1-1}}}{X^{J+\mathbb{I}}} & \overset{\cdot F_{\mathcal{H}}}{\to} & \displaystyle\bigoplus_{\substack{I_1 \dashv \cdots \dashv I_{m_1} \\ |I_i|=d \ \forall i \\ |J|=m_2-N-1}} \mathbb{C} \cdot \frac{\alpha_{I_1} \cdots \alpha_{I_{m_1}}}{X^{J+\mathbb{I}}} \\ \dfrac{\alpha_{I_1} \cdots \alpha_{I_{m_1-1}}}{X^{J+\mathbb{I}}} & \mapsto & \displaystyle\sum_{|K|=d} \frac{\alpha_{I_1} \cdots \alpha_{I_{m_1-1}} \alpha_K}{X^{J-K+\mathbb{I}}}, \end{array}$$

where $\frac{1}{X^{J-K+\mathbb{I}}} = 0$ when $J \ngeq K$. If we write this in the more linear algebraic language, letting $\ell := m_1 - 1$, $k := -m_2 - N - 1$ to simplify notation, we obtain the map

$$\begin{array}{ccc} S^\ell S^d V^* \otimes S^{k+d} V^* & \overset{\sum_K \cdot e_K^* \otimes \lrcorner e_K}{\longrightarrow} & S^{\ell+1} S^d V^* \otimes S^k V^* \\ e_{I_1}^* \cdots e_{I_\ell}^* \otimes e_J^* & \mapsto & \displaystyle\sum_{|K|=d} e_{I_1}^* \cdots e_{I_\ell}^* e_K^* \otimes e_J^* \lrcorner e_K. \end{array} \tag{9.3}$$

where $\lrcorner e_K$ denotes the contraction map. That is,

$$e^*_J \lrcorner e_K := \begin{cases} e^*_{J-K} & \textit{if} \quad J \geqslant K \\ 0 & \textit{if} \quad J \ngeq K \end{cases}$$

When $m_1 \leqslant -N-1$ and $m_2 \geqslant d$ a similar study for

$$H^{N_d}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m_1-1, m_2-d)) \to H^{N_d}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m_1-1, m_2-d))$$

yields a map

$$\begin{array}{ccc} S^{\ell+1}S^dV \otimes S^kV & \stackrel{\sum \lrcorner e^*_K \otimes e_K}{\to} & S^\ell S^dV \otimes S^{k+d}V \\ e_{I_1} \cdots e_{I_{\ell+1}} \otimes e_J & \mapsto & \displaystyle\sum_{|K|=d} (e_{I_1} \cdots e_{I_{\ell+1}}) \lrcorner e^*_K \otimes e_{J+K} \end{array} \tag{9.4}$$

where $\ell := -m_1 - N_d - 1$, $k := m_2 - d$ and where

$$(e_{I_1} \cdots e_{I_{\ell+1}}) \lrcorner e^*_K := \begin{cases} e_{I_1} \cdots \hat{e}_{I_j} \cdots e_{I_{\ell+1}} & \text{if } I_j = K \text{ for some j} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 9.4.1. Observe that the map (9.4) is dual to the map (9.3), therefore as long as we are concerned only with the rank of those map, one might as well only look at application (9.3), which is what we will do from now on.

9.5 The case $N = 1$

When the dimension of V is 2 then the combinatorics is somewhat simpler, and we are able to prove that the maps that are occurring are of maximal rank. Since the exact sequences are slightly different in the case $N = 1$ we write them down.

$$\begin{array}{l} 0 \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \ \to \ H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to \\ \quad \to H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \ \stackrel{a}{\to} \ H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^1(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0 \end{array}$$

$$\begin{array}{lcl} 0 \to H^{N_d-1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) & \to & H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \stackrel{b}{\to} H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to \\ \quad \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) & \to & H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \to H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to 0. \end{array}$$

Our aim is to prove that for each (m_1, m_2) there is at most one non vanishing cohomology group $H^i(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2))$, and that this group sits in a short exact sequence. From this one could at least determine the dimension of these cohomology groups. Using Corollary 9.2.2 we see that for a complete description there are 13 cases to consider.

1. The 4 main cases:

   Case 1: If $m_1 \geqslant 1$ and $m_2 \geqslant d$ then we have only the short exact sequence

   $$0 \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0.$$

   Case 2: If $m_1 \leqslant -N_d - 1$ and $m_2 \leqslant -2$ then we have only the short exact sequence

   $$0 \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1-1, m_2-d)) \to H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to 0.$$

Case 3: If $m_1 \geqslant 1$ and $m_2 \leqslant -2$ then we only have the long exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) &\to H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \\ &\xrightarrow{a} H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^1(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0. \end{aligned}$$

Case 4: If $m_1 \leqslant -N_d - 1$ and $m_2 \geqslant d$ then we only have the long exact sequence

$$\begin{aligned} 0 \to H^{N_d - 1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) &\to H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \\ &\xrightarrow{b} H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0. \end{aligned}$$

2. The 2 overlapping cases:

Case 5: If $m_1 \geqslant 1$ and $0 \leqslant m_2 \leqslant d-2$ then we only have the short exact sequence

$$0 \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to 0.$$

Case 6: If $m_1 \leq -N_d - 1$ and $0 \leqslant m_2 \leqslant d-2$ then we only have the short exact sequence

$$0 \to H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to 0.$$

3. The 6 simple cases:

Case 7: If $m_1 = 0$ and $m_2 \geqslant 0$ or if $m_1 \geqslant 0$ and $m_2 = d-1$ then we only obtain an isomorphism

$$0 \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0.$$

Case 8: If $m_1 \geqslant 1$ and $m_1 = -1$ then we only have the isomorphism

$$0 \to H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to 0.$$

Case 9: If $m_1 = 0$ and $m_2 \leqslant -2$ then we only have the isomorphism

$$0 \to H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^1(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0.$$

Case 10: If $m_1 \leqslant -N_d$ and $m_2 = -1$ or if $m_1 = -N_d$ and $m_2 \leqslant d-2$ then we only have the isomorphism

$$0 \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^{N_d+1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to 0.$$

Case 11: If $m_1 \leq -N_d - 1$ and $m_2 = d-1$ then we only have an isomorphism

$$0 \to H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1, m_2)) \to H^{N_d}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to 0.$$

Case 12: If $m_1 = -N_d$ and $m_2 \geqslant d$ then we only have the isomorphism

$$0 \to H^{N_d - 1}(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(m_1, m_2)) \to H^{N_d}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m_1 - 1, m_2 - d)) \to 0.$$

4. The trivial case:

Case 13: If (m_1, m_2) doesn't belong to one of the above cases then all cohomology groups vanish.

Therefore except in the case 3 and 4 our claim holds. To prove case 3 and 4 we have exactly to prove that the maps a and b are of maximal rank. With what we said in the previous section we have only to study the map :

$$\begin{array}{rcl} S^\ell S^d V^* \otimes S^{k+d} V^* & \stackrel{\Theta_d^{\ell,k}}{\to} & S^{\ell+1} S^d V^* \otimes S^k V^* \\ e_K^* \otimes e_L^* & \mapsto & \displaystyle\sum_{|I|=d} e_K^* \cdot e_I^* \otimes e_L^* \lrcorner e_I. \end{array}$$

We easily see that $\dim\left(S^\ell S^d V^* \otimes S^{k+d} V^*\right) = \binom{\ell+d}{d} \cdot (k+d+1)$ and that $\dim\left(S^{\ell+1} S^d V^* \otimes S^k V^*\right) = \binom{\ell+d+1}{d} \cdot (k+1)$.

Therefore,

$$\begin{array}{rcl} \dim\left(S^\ell S^d V^* \otimes S^{k+d} V^*\right) & \leq & \dim\left(S^{\ell+1} S^d V^* \otimes S^k V^*\right) \\ \Leftrightarrow \quad \dfrac{(\ell+d)!}{d!\ell!}(k+d+1) & \leq & \dfrac{(\ell+d+1)!}{d!(\ell+1)!}(k+1) \\ \Leftrightarrow \quad (\ell+1)(k+d+1) & \leq & (\ell+d+1)(k+1) \\ \Leftrightarrow \quad \ell & \leq & k \end{array}$$

Therefore our question is reduced to prove the following:

Proposition 9.5.1. *With the above notation,*

- *If* $\ell \leqslant k$ *then* $\Theta_d^{\ell,k}$ *is injective.*
- *If* $\ell \geqslant k$ *then* $\Theta_d^{\ell,k}$ *is surjective.*

The rest of this chapter is devoted to the proof of this proposition.

Order on the basis.

We consider a basis (e_0, e_1) for V and the dual basis (e_0^*, e_1^*) on V^*. For any $m \in \mathbb{N}$, this yields a basis $(e_0^{m-i} \cdot e_1^i)_{0 \leqslant i \leqslant m}$ on $S^m V$, and a dual basis $(e_0^{*m-i} \cdot e_1^{*i})_{0 \leqslant i \leqslant m}$ on $S^m V^*$. Set $e_{(m-i,i)} := e_0^{m-i} \cdot e_1^i$ and $e_{(m-i,i)}^* := e_0^{*m-i} \cdot e_1^{*i}$. We have an order on the couples $(m-i, i)$ for $0 \leqslant i \leqslant m$. Just set $(m-i,i) \dashv (m-j,j)$if and only if $i \leqslant j$. This yields in particular a basis for $S^\ell S^d V^*$ whose elements are exactly the ones of the form $e_{I_1}^* \cdot e_{I_2}^* \cdots e_{I_\ell}^*$ with $I_1 \dashv I_2 \dashv \cdots \dashv I_\ell$, where $I_j := (d - i_j, i_j)$. Moreover we can use the order $\dashv$ to induce the lexicographic order on this basis. When we write a matrix representing some linear application between vector spaces of the above form, then we always use the above basis ordered as above.

A warm up.

Even in this case, our proof gets notationally confusing. Therefore we will treat in some details the cases $\ell = 0$ and $\ell = 1$ to help the reader to understand the strategy of the proof, which is just a Gauss algorithm.

Example 1. $\ell = 0$, $d = 2$, $k = 2$. In this case the matrix is just

$$\Theta_2^{0,2} = \left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \hline & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{array}\right)$$

We don't write the 0's for readability. This matrix is certainly injective.

More generally, consider the map $\lrcorner e_{(d-i,i)} : S^{k+d}V^* \to S^k V^*$. The corresponding matrix is

$$\lrcorner e_{(d-i,i)} = \begin{array}{cc} \begin{array}{ccccccccc} 0 & & & i & & i+k & & & d+k \\ \downarrow & & & \downarrow & & \downarrow & & & \downarrow \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \leftarrow 0 \\ \\ \leftarrow k \end{array} \end{array}$$

Now the map corresponding to $\Theta_d^{0,k} : S^{k+d}V^* \to S^d V \otimes S^k V^*$ is just

$$\Theta_d^{0,k} = \left(\begin{array}{c} \lrcorner e_{(d,0)} \\ \hline \lrcorner e_{(d-1,1)} \\ \hline \vdots \\ \hline \lrcorner e_{(0,d)} \end{array} \right)$$

As in the example, using the explicit description of $\lrcorner e_{(d-i,i)}$ it is easy to see that this matrix is injective.

Let us now look at the case when $\ell = 1$ again we start with an example.

Example 2.

$$\Theta_2^{1,2} = \left(\begin{array}{ccccc|ccccc|ccccc}
1 & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & \\
\hline
 & 1 & & & & 1 & & & & & & & & & \\
 & & 1 & & & & 1 & & & & & & & & \\
 & & & 1 & & & & 1 & & & & & & & \\
\hline
 & & 1 & & & & & & & & 1 & & & & \\
 & & & 1 & & & & & & & & 1 & & & \\
 & & & & 1 & & & & & & & & 1 & & \\
\hline
 & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
\hline
 & & & & & & & 1 & & & & 1 & & & \\
 & & & & & & & & 1 & & & & 1 & & \\
 & & & & & & & & & 1 & & & & 1 & \\
\hline
 & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & 1
\end{array} \right)$$

Again a Gauss algorithm proves that this matrix is injective.

More generally for $\ell = 1$ and any k, d we have ,

$$\Theta_d^{1,k} = \begin{pmatrix}
\lrcorner e_{(d,0)} & & & & & \\
\lrcorner e_{(d-1,1)} & \lrcorner e_{(d,0)} & & & & \\
\lrcorner e_{(d-2,2)} & & \lrcorner e_{(d,0)} & & & \\
\vdots & & & \ddots & & \\
\vdots & & & & \ddots & \\
\lrcorner e_{(0,d)} & & & & & \lrcorner e_{(d,0)} \\
& \lrcorner e_{(d-1,1)} & & & & \\
& \lrcorner e_{(d-2,2)} & \lrcorner e_{(d-1,1)} & & & \\
& \vdots & & \ddots & & \\
& \vdots & & & \ddots & \\
& \lrcorner e_{(0,d)} & & & & \lrcorner e_{(d-1,1)} \\
& & \lrcorner e_{(d-2,2)} & & & \\
& & \lrcorner e_{(d-3,3)} & \lrcorner e_{(d-2,2)} & & \\
& & \vdots & & \ddots & \\
& & \vdots & & \vdots & \vdots \\
& & \vdots & & & \vdots \\
& & & & \lrcorner e_{(1,d-1)} & \\
& & & & \lrcorner e_{(0,d)} & \lrcorner e_{(1,d-1)} \\
& & & & & \lrcorner e_{(0,d)}
\end{pmatrix}$$

Now suppose $k \geqslant 1$. We are going to prove that $\Theta_d^{1,k}$ is injective. The first remark to make is that in the matrix $\Theta_d^{1,k}$ one retrieves the matrix $\Theta_d^{0,k}$ in the upper left corner. More precisely, one can write,

$$\Theta_d^{1,k} = \left(\begin{array}{c|c} \Theta_d^{0,k} & M_d \\ \hline 0 & \widetilde{\Theta}_{d-1}^{1} \end{array}\right)$$

For some matrix M_d and $\widetilde{\Theta}_{d-1}^{1}$. Now the idea is to use some sort of an induction hypothesis combined with the injectivity properties of $\Theta_d^{0,k}$. The thing we really will be using is that the matrix $\Theta_d^{0,k}$ has some "extra injectivity" that one can exploit.

Let us make this approach more precise. Fix $k \geqslant 1$, $d \geqslant 0$ and $\ell = 1$. We decompose the matrix $\Theta_d^{1,k}$ according to the "big" blocks, therefore we let

$$\widetilde{\Theta}_0^{1} := \boxed{\lrcorner e_{(0,d)}}$$

$$\widetilde{\Theta}_1^{1} := \begin{array}{|c|c|} \hline \lrcorner e_{(1,d-1)} & \\ \hline \lrcorner e_{(0,d)} & \lrcorner e_{(1,d-1)} \\ \hline & \lrcorner e_{(0,d)} \\ \hline \end{array}$$

$$\vdots$$

$$\widetilde{\Theta}_d^{1} := \Theta_d^{1,k}.$$

We also let

$$\widetilde{\Theta}_0^{0} := \widetilde{\Theta}_0^{1} = \boxed{\lrcorner e_{(0,d)}}$$

$$\widetilde{\Theta}_1^{0} := \begin{array}{|c|} \hline \lrcorner e_{(1,d-1)} \\ \hline \lrcorner e_{(0,d)} \\ \hline \end{array}$$

$$\vdots$$

$$\widetilde{\Theta}_d^0 := \Theta_d^{0,k}.$$

Moreover we let

$$M_1 := \begin{array}{|c|}\hline 0 \\ \hline \lrcorner e_{(1,d-1)} \\ \hline \end{array}$$

$$M_2 := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \lrcorner e_{(2,d-2)} & 0 \\ \hline 0 & \lrcorner e_{(2,d-2)} \\ \hline \end{array}$$

$$\vdots$$

$$M_d := \begin{array}{|c|ccc|c|}\hline 0 & \cdots & & \cdots & 0 \\ \hline \lrcorner e_{(d,0)} & & & & \\ \cline{1-2} 0 & \multicolumn{1}{c|}{\lrcorner e_{(d,0)}} & & & \\ \cline{1-2} \vdots & & \ddots & & \\ \cline{1-1} \cline{5-5} 0 & & & & \lrcorner e_{(d,0)} \\ \hline \end{array}$$

We need one more notation.

$$N_i := \begin{array}{c}\begin{array}{ccccccc} 0 & & & i & & & d+k \\ \downarrow & & & \downarrow & & & \downarrow \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{array}{l} \leftarrow 0 \\ \\ \leftarrow k \end{array}\end{array}$$

the matrix where the only non zero term is on line 0 and column i. And then we let

$$M_1' := \begin{array}{|c|}\hline 0 \\ \hline N_{d-1} \\ \hline \end{array}$$

$$M_2' := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline N_{d-2} & 0 \\ \hline 0 & N_{d-2} \\ \hline \end{array}$$

$$\vdots$$

$$M_d := \begin{array}{|c|ccc|c|}\hline 0 & \cdots & & \cdots & 0 \\ \hline N_0 & & & & \\ \cline{1-2} 0 & \multicolumn{1}{c|}{N_0} & & & \\ \cline{1-2} \vdots & & \ddots & & \\ \cline{1-1} \cline{5-5} 0 & & & & N_0 \\ \hline \end{array}$$

Therefore, for all $0 < i \leqslant d$ one has

$$\widetilde{\Theta}_i^1 = \left(\begin{array}{c|c} \widetilde{\Theta}_i^0 & M_i \\ \hline 0 & \widetilde{\Theta}_{i-1}^i \end{array} \right)$$

Certainly $\widetilde{\Theta}_i^1$ will not be injective for $1 < d$ since the matrix contains columns of zero's. However we want to prove that those zero's are the only obstructions to injectivity. This motivates the following (provisional) definition (see Definition 9.5.3).

Definition 9.5.2. Let E be an r-dimensional complex vector space with a basis $(\epsilon_1, \cdots, \epsilon_r)$, and F an other complex vector space. Let $f : E \to F$ be a linear application. Set,

$$E_f^\epsilon := \mathrm{Span}\{\epsilon_i \ / \ f(\epsilon_i) \neq 0\}.$$

We say that f is *almost injective* if $f_{|E_f^\epsilon}$ is injective

We will now prove by induction that

Hypothesis H_1. $\widetilde{\Theta}_i^1$ is almost injective for all $0 \leqslant i \leqslant d$ for the chosen bases.

This will indeed prove that $\Theta_d^1 = \widetilde{\Theta}_d^1$ is injective. An obviously, H_1 holds for $i = 0$. Let us now prove H_1. Fix $i \geqslant 1$. Using H_1 for $i - 1$ combined with a Gauss algorithm we get,

$$\left(\begin{array}{c|c} \widetilde{\Theta}_i^0 & M_i \\ \hline 0 & \widetilde{\Theta}_{i-1}^1 \end{array}\right) \overset{\text{Gauss}}{\rightsquigarrow} \left(\begin{array}{c|c} \widetilde{\Theta}_i^0 & M_i' \\ \hline 0 & \widetilde{\Theta}_{i-1}^1 \end{array}\right)$$

To see that this holds, one just have to look carefully at the exact expression of the appearing matrices. To finish the proof we then just have to prove that, the matrix

$$\left(\begin{array}{c|c} \widetilde{\Theta}_i^0 & M_i' \end{array}\right)$$

is almost injective. This is again an induction on the following hypothesis.

Hypothesis H_2. $\left(\begin{array}{c|c} \widetilde{\Theta}_i^0 & M_i' \end{array}\right)$ is almost injective for all $1 \leqslant i \leqslant d$.

It is important to note that for H_2 to hold one needs $k \geqslant 1$. It is now easy to check H_1 and H_2, and therefore we conclude that $\Theta_d^{1,k}$ is injective for $k \geqslant \ell = 1$.

We would like to point out the rough ideas we will use to generalize this to $k \geqslant \ell \geqslant 1$. The idea is to write $\Theta_d^{\ell,k}$ as a matrix involving:

- $\Theta_d^{\ell-1,k}$.
- Some sort of a diagonal matrix.
- A matrix similar to $\Theta_d^{\ell,k}$ but smaller.

Once one has this decomposition one only has to find out what the relevant induction hypothesis is. Morally, the hypothesis $\ell \leqslant k$ comes from the necessity to fully exploit the injectivity of $\Theta_d^{0,k}$. For the injectivity of $\Theta_d^{1,k}$ we need to be able to get informations from $\Theta_d^{0,k}$ if we added some constrain of "order 1", for the injectivity of $\Theta_d^{\ell,k}$ we will need to add to $\Theta_d^{0,k}$ a constraint of "order ℓ", and this is exactly where the hypothesis $\ell \geqslant k$ is needed. The proof is now just a matter of notation to make this precise.

Notation.

For $0 \leqslant i \leqslant d$ we set

$$
\begin{array}{rcl}
W_i^{\ell,d} & := & \text{Span}\left(e^*_{I_1}e^*_{I_2}\cdots e^*_{I_\ell} \ \ / \ \ (d-i,i) \dashv I_1 \dashv \cdots \dashv I_\ell\right) \subseteq S^\ell S^d V^* \\
A_i^{\ell,d} & := & \text{Span}\left(e^*_{(d-i,i)}e^*_{I_2}\cdots e^*_{I_\ell} \ \ / \ \ (d-i,i) = I_1 \dashv \cdots \dashv I_\ell\right) \subseteq W_i^{\ell,d} \\
T_i^{k+d} & := & \text{Span}\left(e^*_{(k+d-p,p)} \ \ / \ \ i \leqslant p\right) \subseteq S^{k+d}V^* \\
T_{i,j}^{k+d} & := & \text{Span}\left(e_{(k+d-p,p)} \ \ / \ \ i \leqslant p \leqslant i+j\right) \subseteq T_i^{k+d} \\
Q_j^k & := & \text{Span}\left(e^*_{(k-q,q)} \ \ / \ \ q \leqslant j\right) \subseteq S^k V^*.
\end{array}
$$

Observe that $W_i^{\ell,d} = A_i^{\ell,d} \oplus W_{i+1}^{\ell,d}$. Note also that we have an isomorphism

$$
\begin{array}{rcl}
W_i^{\ell-1,d} & \to & A_i^{\ell,d} \\
e_{L_2}\cdots e_{L_\ell} & \mapsto & e_{(d-i,i)}e_{L_2}\cdots e_{L_\ell}
\end{array}
$$

We also define,

$$
\begin{array}{rcl}
\Theta_i^{\ell,k,d} : W_i^{\ell,d} \otimes S^{k+d}V^* & \to & W_i^{\ell+1,d} \otimes S^k V^* \\
e^*_{I_1}\cdots e^*_{I_\ell} \otimes e^*_{(k+d-p,p)} & \mapsto & \displaystyle\sum_{i \leqslant j \leqslant d} e^*_{I_1}\cdots e^*_{I_\ell} e^*_{(d-j,j)} \otimes e^*_{(k+d-p,p)} \lrcorner e_{(d-j,j)}.
\end{array}
$$

Moreover if $0 \leqslant j \leqslant k$ we let

$$
\begin{array}{rcl}
\Phi_{i,j}^{\ell,k,d} : W_{i+1}^{\ell,d} \otimes S^{k+d}V^* & \to & W_{i+1}^{\ell,d} \otimes S^k V^* \\
e^*_{L1}\cdots e^*_{L_\ell} \otimes e^*_{(k+d-p,p)} & \mapsto & \begin{cases} e^*_{L1}\cdots e^*_{L_\ell} \otimes e^*_{(k+d-p,p)} \lrcorner e_{(d-i,i)} & \textit{if } \ p \leqslant j \\ \\ 0 & \textit{otherwise} \end{cases}
\end{array}
$$

Observe that $\Phi_{i,k}^{\ell,k,d} = Id \otimes \bullet \lrcorner e^{\leqslant j}_{(d-i,i)}$. Where

$$
\begin{array}{rcl}
\lrcorner e^{\leqslant j}_{(d-i,i)} : S^{k+d}V^* & \to & S^k V^* \\
e^*_{(k+d-p,p)} & \mapsto & \begin{cases} e^*_{(k+d-p,p)} \lrcorner e_{(d-i,i)} & \textit{if } \ p \leqslant j \\ \\ 0 & \textit{otherwise} \end{cases}
\end{array}
$$

In matricial notation we obtain

$$
\lrcorner e^{\leqslant j}_{(d-i,i)} = \begin{array}{cc}
\begin{array}{ccccccccc}
0 & & & i & & i+j-1 & & & d+k \\
\downarrow & & & \downarrow & & \downarrow & & & \downarrow
\end{array} & \\
\begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & & \vdots & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix} &
\begin{array}{c}
\leftarrow 0 \\ \\ \leftarrow j-1 \\ \\ \leftarrow k
\end{array}
\end{array}
$$

And

$$
\Phi_{i,k}^{\ell,k,d} := \begin{array}{|c|c|c|c|}
\hline
\lrcorner e_{(d,0)} & 0 & \cdots & 0 \\
\hline
0 & \lrcorner e_{(d,0)} & & \\
\hline
\vdots & & \ddots & \\
\hline
0 & & & \lrcorner e_{(d,0)} \\
\hline
\end{array}
$$

We have the identification

$$W_i^{\ell,d} = A_i^{\ell,d} \oplus W_{i+1}^{\ell,d} = W_i^{\ell-1,d} \oplus W_{i+1}^{\ell,d}$$

and similarly, the identification

$$W_i^{\ell+1,d} = W_i^{\ell-1,d} \oplus W_{i+1}^{\ell,d} \oplus W_{i+1}^{\ell+1,d}.$$

Under those identifications, we have an inductive representation of $\Theta_i^{\ell,k,d}$,

$$\left(\begin{array}{c|c} \Theta_i^{\ell-1,k,d} & \begin{array}{c} 0 \\ \hline \Phi_{i,k}^{\ell,k,d} \end{array} \\ \hline 0 & \Theta_{i+1}^{\ell,k,d} \end{array}\right)$$

Let us also present a notion stronger than injectivity that will come in handy during the proof. Let F be a complex vector space with a basis $(u_1, \dots, u_m)$ take $1 \leqslant r \leqslant m$ and let $F_r := \bigoplus_{i=1}^r \mathbb{C} \cdot u_i$. Take a vector space E a linear map $f : F \to E$.

Definition 9.5.3. We say that f is *freely-injective* on F_r if for any ξ_i, $1 \leqslant i \leqslant m$,

$$\sum_{i=1}^m \xi_i u_i = 0 \Rightarrow \xi_1 = \cdots = \xi_r = 0.$$

Clearly if f is freely-injective on F_r then $f_{|F_r}$ if injective, the converse is however not true as it is easy to see. Let us make an elementary remark.

Proposition 9.5.4. *If $f : F \to E$. Then f is freely-injective on F_r if and only if the matrix representing f can be transformed using only elementary operations on the lines in a matrix of the form,*

$$\begin{array}{c} \begin{array}{cccccc} 1 & & r & & & \\ \downarrow & & \downarrow & & & \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & 1 & 0 & \cdots & 0 \\ & & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix} \end{array}$$

Let us make another elementary remark. Let F and E are two vector spaces. Consider a map

$$f : F \otimes S^{k+d} V^* \to E \otimes S^k V^*$$

which corresponds to a block matrix M. Then $Im(f) \subseteq E \otimes Q_j^k$ if and only if each block appearing in M is of the form

$$\begin{array}{cc} \begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow \quad j+1 \\ \\ \\ \end{array} \end{array}$$

The case $\ell \leqslant k$

With those notation, we are ready to prove our claim. Suppose now that $\ell \leqslant k$ we will prove that $\Theta^{\ell,k,d}$ is injective. We will do this by induction on ℓ and i, with the following hypothesis.

Fix any $j' \leqslant j \leqslant k-\ell-1$.

Hypothesis H. Under the identification $W_i^{\ell+1,d} \otimes S^kV^* \equiv \left(W_i^{\ell,d} \oplus W_{i+1}^{\ell+1,d}\right) \otimes S^kV^*$. For any vector space F and for any linear map $\Gamma_{j'} : F \otimes S^{k+d}V^* \to \left(W_i^{\ell,d} \oplus W_{i+1}^{\ell+1,d}\right) \otimes S^kV^*$ satisfying $Im(\Gamma_{j'}) \subseteq \left(W_i^{\ell,d} \oplus W_{i+1}^{\ell+1,d}\right) \otimes Q_j^k$ the application

$$\left(W_i^{\ell,d} \oplus W_{i+1}^{\ell+1,d} \oplus F\right) \otimes S^{k+d}V^* \quad \to \quad \left(W_i^{\ell,d} \oplus W_{i+1}^{\ell+1,d}\right) \otimes S^kV^*$$

associated to the matrix

$$\left(\begin{array}{c|c|c} \Theta_i^{\ell,k,d} & \begin{array}{c} 0 \\ \hline \Phi_{i,j}^{\ell+1,k,d} \end{array} & \Gamma_{j'} \end{array}\right)$$

is freely-injective on $\left(W_i^{\ell,d} \otimes T_{i+j'+1}^{k+d}\right) \oplus \left(W_{i+1}^{\ell+1,d} \otimes T_{i+j'+1,j-j'}^{k+d}\right)$.

If this hypothesis is verified the result will follow from the special case $j = j' = -1$ and $i = 0$.
To do so we will make an induction on ℓ from 0 to k and an induction on i from d to 0.

Initializing $i = d$.

This is easily verified after remarking that the application $\Theta_d^{\ell,k,d}$ can be identified with the contraction application,

$$\begin{array}{rcl} \Theta_d^{\ell,k,d} \cong \lrcorner e_{(0,d)} : S^{k+d}V^* & \to & S^kV^* \\ e^*_{(k+d-p,p)} & \mapsto & e^*_{(k+d-p,p)} \lrcorner e_{(0,d)}. \end{array}$$

Initializing $\ell = 0$.

We start by checking hypothesis H when $\ell = 0$. We suppose therefore that $j \leqslant k-1$.
The application

$$\left(\begin{array}{c|c|c} \Theta_i^{0,k,d} & \begin{array}{c} 0 \\ \hline \Phi_{i,j}^{1,k,d} \end{array} & \Gamma_{j'} \end{array}\right)$$

can be viewed as the matrix

$$\left(\begin{array}{c|c|c|c|c} \lrcorner e_i & & \cdots \quad \cdots & & \Gamma^i_{j'} \\ \hline \lrcorner e_{i+1} & \lrcorner e_i^{\leqslant j} & & & \Gamma^{i+1}_{j'} \\ \hline \vdots & & \ddots & & \vdots \\ \vdots & & \quad \ddots & & \vdots \\ \hline \lrcorner e_d & & & \lrcorner e_i^{\leqslant j} & \Gamma^d_{j'} \end{array}\right).$$

It is now easy to conclude, by a gauss algorithm, using the description we made above of the matrix $\lrcorner e_i^{\leqslant j}$.

Checking hypothesis H inductively.

The matrix

$$\left(\begin{array}{c|c|c} \Theta_i^{\ell,k,d} & \begin{array}{c} 0 \\ \hline \Phi_{i,j}^{\ell+1,k,d} \end{array} & \Gamma_{j'} \end{array}\right)$$

can be rewritten

$$\left(\begin{array}{c|c|c|c}
 & 0 & 0 & \\
\cline{2-3}
\Theta_i^{\ell-1,k,d} & & & \Gamma^1_{j'} \\
 & \Phi_{i,k}^{\ell,k,d} & 0 & \\
\hline
0 & \Theta_{i+1}^{\ell,k,d} & \Phi_{i,j}^{\ell+1,k,d} & \Gamma^2_{j'}
\end{array}\right).$$

We start be applying our hypothesis H to the matrix

$$\left(\begin{array}{c|c|c} \Theta_{i+1}^{\ell,k,d} & \Phi_{i,j}^{\ell+1,k,d} & \Gamma^2_{j'} \end{array} \right).$$

Viewed as an application from $\left(W_{i+1}^{\ell,d} \oplus W_{i+1}^{\ell+1,d} \oplus F\right) \otimes S^{k+d}V^*$ to $W_{i+1}^{\ell+1,d} \otimes S^kV^*$. Using H, this map is freely-injective on $W_{i+1}^{\ell,d} \otimes T_{i+j+2}^{k+d}$. Now as in Proposition 9.5.4, by elementary operations on the lines we can transform this matrix into the following:

$$\left(\begin{array}{c|c|c} \widetilde{\Theta}_{i+1}^{\ell,k,d} & \Phi_{i,j}^{\ell+1,k,d} & \Gamma^2_{j'} \end{array} \right),$$

where all the blocs appearing in $\widetilde{\Theta}_{i+1}^{\ell,k,d}$ are of the form,

$$\begin{array}{cc}
\left(\begin{array}{ccc|cccccc}
* & \cdots & * & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & \vdots & \vdots & & & & & \vdots \\
* & \cdots & * & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \cdots & 0 & * & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & * & 0 & \cdots & 0
\end{array}\right)
&
\begin{array}{l}
\\ \\ \\
\leftarrow \quad j+1 \\
\\ \\
\end{array}
\end{array}$$

By free-injectivity on $W_{i+1}^{\ell,d} \otimes T_{i+j+2}^{k+d}$ we can, by elementary operations on lines, clear out some terms in the upper matrix to transform it into the matrix

$$\left(\begin{array}{c|c|c|c}
 & 0 & 0 & \\
\cline{2-3}
\Theta_i^{\ell-1,k,d} & & & \Gamma^1_{j'} \\
 & \Phi_{i,j+1}^{\ell,k,d} & 0 & \\
\hline
0 & \widetilde{\Theta}_{i+1}^{\ell,k,d} & \Phi_{i,j}^{\ell+1,k,d} & \Gamma^2_{j'}
\end{array}\right).$$

Now we apply hypothesis H to the matrix

$$\left(\begin{array}{c|c|c|c}
 & 0 & 0 & \\
\cline{2-3}
\Theta_i^{\ell-1,k,d} & & & \Gamma^1_{j'} \\
 & \Phi_{i,j+1}^{\ell,k,d} & 0 &
\end{array}\right).$$

Our induction hypothesis tells us that it is freely-injective on $\left(W_i^{\ell-1,d} \otimes T^{k+d}_{i+j'+1}\right) \oplus \left(W^{\ell,d}_{i+1} \otimes T^{k+d}_{i+j'+1,j-j'+1}\right)$. Again we use elementary operations on the lines and free-injectivity to clear out some more terms, to get now the matrix

$$\left(\begin{array}{c|c|c|c}
 & 0 & 0 & \\ \cline{2-3}
\Theta_i^{\ell-1,k,d} & & & \Gamma^1_{j'} \\
 & \Phi^{\ell,k,d}_{i,j+1} & 0 & \\ \hline
0 & \widehat{\Theta}^{\ell,k,d}_{i+1} & \Phi^{\ell+1,k,d}_{i,j} & \Gamma^2_{j'}
\end{array}\right),$$

where each one of the blocks of $\widehat{\Theta}^{\ell,k,d}_{i+1}$ is of the form

$$\begin{array}{cc}
\left(\begin{array}{ccc|cccccc}
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & \vdots & \vdots & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hline
0 & \cdots & 0 & * & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & * & 0 & \cdots & 0
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \leftarrow \quad j+1 \\ \\ \\ \end{array}
\end{array}$$

Thus every row of blocks of the matrix

$$\left(\begin{array}{c|c|c} \widehat{\Theta}^{\ell,k,d}_{i+1} & \Phi^{\ell+1,k,d}_{i,j} & \Gamma^2_{j'} \end{array} \right)$$

is of the form

$$\begin{array}{cc}
\left(\begin{array}{ccccccc|ccccc|ccc}
0 & \cdots & & & & \cdots & 0 & 0 & \cdots & & \cdots & 0 & * & \cdots & * \\
\vdots & & & & & & \vdots & \vdots & & & & \vdots & \vdots & & \vdots \\
0 & \cdots & & & & \cdots & 0 & 0 & \cdots & & \cdots & 0 & * & \cdots & * \\ \hline
0 & \cdots & & & & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & & & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & & & & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \hline
* & \cdots & 0 & \cdots & * & \cdots & 0 & 0 & \cdots & & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \vdots & & & & \vdots & \vdots & & \vdots \\
0 & \cdots & * & \cdots & 0 & \cdots & * & 0 & \cdots & & \cdots & 0 & 0 & \cdots & 0
\end{array}\right) &
\begin{array}{c} \\ \\ \leftarrow \quad j' \\ \\ \\ \\ \leftarrow \quad j+1 \\ \\ \\ \end{array}
\end{array}$$

From this we obtain that free-injectivity on $\left(W^{\ell,d}_{i+1}\right) \otimes T^{k+d}_{i+j'+1,j-j'}$. And combining all this we obtain, at last, the desired result.

The case $\ell \geqslant k$

This case is easily deduce from what we already prove. We will prove the following hypothesis,

Hypothesis H' For all $d \geqslant i \geqslant 0$ and all $\ell \geqslant k$ the application $\Theta_i^{\ell,k,d}$ is surjective.

If hypothesis H' holds then we conclude with the special case $i = 0$. Let us now prove hypothesis H' by induction on i from d to 0 and induction on $\ell \geqslant k$.

Initializing $i = d$.

If $i = d$ this is clear since $\Theta_d^{\ell,k,d} \cong \lrcorner e_{(0,d)}$ is just a contraction morphism, which is surjective.

Initializing $\ell = k$.

This is reduced to a dimension count. Thanks to hypothesis H we know that, for all $0 \leqslant i \leqslant d$, $\Theta_i^{k,k,d}$ is injective when restricted to $W_i^{k,d} \otimes T_i^{k+d}$. Therefore to see that $\Theta_i^{k,k,d}$ is surjective it suffices to prove that

$$\dim\left(W_i^{k,d} \otimes T_i^{k+d}\right) = \dim\left(W_i^{k+1,d} \otimes S^k V^*\right)$$

But we have, for any ℓ', k',

$$\dim\left(W_i^{\ell',d}\right) = \binom{\ell' + d - i}{d - i}$$
$$\dim\left(T_i^{k'+d}\right) = k' + d - i + 1.$$

In our situation we are therefore reduced to check that

$$\binom{k + d - i}{d - i}(k + d - i + 1) = \binom{k + 1 + d - i}{d - i}(k + 1)$$

Which is readily verified.

Checking hypothesis H' inductively.

Take $\ell > k$, $0 \leqslant i < d$ and that H' holds for all $k \leqslant \ell' < \ell$ and all $i' > i$. Under the usual identifications, we rewrite $\Theta_i^{\ell,k,d}$ as

$$\left(\begin{array}{c|c} \Theta_i^{\ell-1,k,d} & \begin{array}{c} 0 \\ \hline \Phi_{i,k}^{\ell,k,d} \end{array} \\ \hline 0 & \Theta_{i+1}^{\ell,k,d} \end{array}\right)$$

Now, the applications $\Theta_{i+1}^{\ell,k,d}$ and $\Theta_i^{\ell-1,k,d}$ are both surjective. Therefore $\Theta_i^{\ell,k,d}$ is surjective as well.

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